

## Homework 12 – due 11/21/07

### Math 600

61. (a) (5 points) Let  $R$  be any commutative ring with identity, and let  $P$  be an  $n \times n$  matrix whose columns span  $R^n$  over  $R$ . Show that  $P$  is invertible in  $M_n(R)$ .

(b) (5 points) With  $R$  as above, suppose  $V = R^n$  is generated by  $n$  elements  $x_1, \dots, x_n$ . Show that these elements form a basis for  $V$ .

62. (a) (10 points) Suppose  $R$  is a PID and  $W \subset V$  are finitely generated torsion  $R$ -modules. Suppose the invariants of  $V$  are  $a_1 | a_2 | \dots | a_r$ , and the invariants of  $W$  are  $b_1 | b_2 | \dots | b_s$ . Show that  $s \leq r$  and that  $b_{s-j} | a_{r-j}$  for each  $j$ .

(b) (5 points) Show that  $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/16\mathbb{Z}$  has no submodule isomorphic to  $\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$ .

63. Let  $R$  be a Noetherian ring (as usual, let's assume it's commutative with identity). Let  $M$  be an  $R$ -module with  $M \neq 0$ . Show that there exists  $x \in M$  such that  $\text{Ann}(x)$  is a prime ideal in  $R$ .

64. Find the structure of the abelian group  $V$  with generators  $x, y, z$  and relations

$$\begin{aligned} -4x + 2y + 6z &= 0 \\ 6x + 2y + 6z &= 0 \\ 7x + 4y + 15z &= 0. \end{aligned}$$

65. Suppose the abelian group  $V$  is presented with  $s$  generators and  $t$  relations, where  $s \geq t$ . Show that “the” free part of  $V$  has rank  $\geq s - t$ .