

## Solutions to Homework 12

Math 600, Fall 2007

**61. (a) (5 points)** Let  $R$  be any commutative ring with identity, and let  $P$  be an  $n \times n$  matrix whose columns span  $R^n$  over  $R$ . Show that  $P$  is invertible in  $M_n(R)$ .

Let  $\{e_1, \dots, e_n\}$  be the standard basis of  $R^n$ . Let  $v_1, \dots, v_n \in R^n$  be such that  $Pv_i = e_i$ . The  $v_i$  exist because the columns of  $P$  span  $R^n$ . Let  $Q$  be the matrix whose columns are  $v_i$ . Then  $PQ = I_n$ , where  $I_n$  is the identity matrix. This shows  $\det(P) \det(Q) = 1$  and hence  $\det(P)$  is a unit. Therefore  $P^{-1}$  exists.

**61. (b) (5 points)** With  $R$  as above, suppose  $V = R^n$  is generated by  $n$  elements  $x_1, \dots, x_n$ . Show that these elements form a basis for  $V$ .

Let  $P$  be the matrix whose columns are  $\{x_1, \dots, x_n\}$ . By hypothesis, the columns of  $P$  span  $R^n$  over  $R$  and hence by part a) we know that  $P$  is invertible. The linear dependence of the set  $\{x_1, \dots, x_n\}$  would contradict the fact that  $P$  is invertible, hence  $\{x_1, \dots, x_n\}$  is a basis for  $R^n$ .

**62. (a) (10 points)** Suppose  $R$  is a PID and  $W \subset V$  are finitely generated torsion  $R$ -modules. Suppose the invariants of  $V$  are  $a_1|a_2|\dots|a_r$ , and the invariants of  $W$  are  $b_1|b_2|\dots|b_s$ . Show that  $s \leq r$  and that  $b_{s-j}|a_{r-j}$  for each  $j$ .

First we prove that for torsion modules  $W \subset V$  the number of invariant factors of  $W$  is less than or equal to that of  $V$ . We have an exact sequence

$$0 \rightarrow R^r \xrightarrow{A} R^r \rightarrow V \rightarrow 0$$

where  $A$  is the diagonal matrix with  $a_1, \dots, a_r$  on the diagonal. Let  $W' \subset R^r$  be the inverse image of  $W$  under  $R^r \rightarrow V$ . Let  $K = \text{im}(A) = \bigoplus_{i=1}^r a_i R$ . Then  $K \subset W'$  is free of rank  $r$ , hence  $W'$  itself is free of rank  $r$  and there is a basis  $\{f_1, \dots, f_r\}$  of  $W'$  such that  $K \subset W'$  has basis  $\{c_1 f_1, \dots, c_r f_r\}$ . The  $c_i \neq 0$ , since  $W'/K = W$  is torsion and they satisfy  $c_1 | c_2 | \dots | c_r$ . Let  $1 \leq \tau \leq r$  be the largest integer such that  $c_1 = c_2 = \dots = c_\tau = 1$ . We set  $s = r - \tau$  and  $b_{s-j} = c_{r-j}$  for  $0 \leq j \leq s - 1$  (If  $\tau = r$  then  $W = 0$ ). Thus we have shown  $s = r - \tau \leq r$ .

Let  $V = \bigoplus_p V_p$  and  $W = \bigoplus_p W_p$  be the primary decomposition of  $V$  and  $W$ . It is easy to show that  $W_p \subset V_p$ . Let  $V_p = R/(p^{n_1}) \oplus \dots \oplus R/(p^{n_k})$  and  $W_p = R/(p^{m_1}) \oplus \dots \oplus R/(p^{m_l})$  with  $n_i \leq n_{i+1}$

and  $m_i \leq m_{i+1}$ . We have already shown above that  $k \geq l$ . Since  $p^{n_k}V_p = 0$  and  $p^{n_k}W_p \subset p^{n_k}V_p$  it follows that  $p^{n_k}W_p = 0$  and hence  $m_l \leq n_k$ . Suppose  $m_{l-j} > n_{k-j}$  for some  $j$ . Let  $1 \leq j \leq l-1$  be the smallest integer such that  $m_{l-j} > n_{k-j}$ . We have

$$p^{n_{k-j}}V_p = \bigoplus_{i=k-j+1}^k R/(p^{n_i-n_{k-j}})$$

which clearly has at most  $j$  invariant factors. On the other we have

$$p^{n_{k-j}}W_p = p^{n_{k-j}}(R/(p^{m_1}) \oplus \dots \oplus R/(p^{m_{l-j-1}})) \bigoplus (R/(p^{m_{l-j}-n_{k-j}}) \oplus \dots \oplus R/(p^{m_l-n_{k-j}}))$$

which has at least  $j+1$  invariant factors. This contradicts the fact that the number of invariant factors of  $p^{n_{k-j}}W_p$  cannot exceed the number of invariant factors of  $p^{n_{k-j}}V_p$ . Thus such a  $j$  does not exist, and hence  $m_{l-j} \leq n_{k-j}$  for each  $j$ . Since this is true for each prime  $p$ , we get  $b_{s-j} | a_{r-j}$  for each  $j$ .

**63 (10 points)** Let  $R$  be a Noetherian ring (as usual, let's assume it's commutative with identity). Let  $M$  be an  $R$ -module with  $M \neq 0$ . Show that there exists  $x \in M$  such that  $\text{Ann}(x)$  is a prime ideal in  $R$ .

Let  $\mathcal{S} = \{\text{Ann}(x) \mid x \in M - \{0\}\}$  be a collection of ideals. Note that  $R \notin \mathcal{S}$  and  $\mathcal{S}$  is nonempty, hence the Noetherian property of  $R$  implies that  $\mathcal{S}$  has a maximal element  $I = \text{Ann}(x)$ . Suppose  $ab \in I$  with  $b \notin I$ , then  $a \in \text{Ann}(bx)$ . But  $\text{Ann}(bx) \supset \text{Ann}(x) = I$ , and therefore the maximality of  $\text{Ann}(x)$  implies  $\text{Ann}(bx) = \text{Ann}(x) = I$  (where we use the fact that  $bx \neq 0$ ). Thus  $a \in \text{Ann}(x)$ , showing that  $I = \text{Ann}(x)$  is a prime ideal.

**64 (10 points)** Find the structure of the abelian group  $V$  with generators  $x, y, z$  and relations

$$\begin{aligned} -4x + 2y + 6z &= 0 \\ 6x + 2y + 6z &= 0 \\ 7x + 4y + 15z &= 0 \end{aligned}$$

Let  $A$  be the coefficient matrix of the above system of linear equations. We have an exact sequence

$$\mathbb{Z}^3 \xrightarrow{A} \mathbb{Z}^3 \rightarrow V \rightarrow 0$$

We know that we can pick new bases for the first and second modules in the above exact sequence so that the matrix  $A$  transforms to  $QAP^{-1} = \text{diag}(a_1, a_2, a_3)$  for some  $Q, P \in GL(3, \mathbb{Z})$ . The abelian

group  $V$  is isomorphic to  $\mathbb{Z}/(a_1) \oplus \mathbb{Z}/(a_2) \oplus \mathbb{Z}/(a_3)$ . Let  $I_m \subset \mathbb{Z}$  be the ideal generated by the  $m \times m$  minor determinants of the matrix  $A$ . We know that the corresponding ideals for  $QAP^{-1}$  are the same (see hw11 solutions), therefore,  $I_1 = (a_1)$ ,  $I_2 = (a_1a_2)$ , and  $I_3 = (a_1a_2a_3)$ . Clearly  $I_1 = (1) = (a_1)$  since  $2, 15 \in I_1$ . Also  $I_3 = \det(A) = (-60) = (a_1a_2a_3)$ . Reducing  $A$  mod 2 we see that all  $2 \times 2$  minor determinants are 0 mod 2 and so  $2 \mid a_1a_2$ . Also  $-2$  is the determinant of the minor obtained by deleting column 3 and row 2, whence  $(a_1a_2) = (2)$ . Thus we can choose  $a_1 = 1$ ,  $a_2 = 2$  and  $a_3 = 30$ , and therefore  $V = \mathbb{Z}/(2) \oplus \mathbb{Z}/(30)$

**65 (10 points)** Suppose the abelian group  $V$  is presented with  $s$  generators and  $t$  relations, where  $s \geq t$ . Show that the free part of  $V$  has rank  $\geq s - t$ .

**Proof 1** We have an exact sequence

$$\mathbb{Z}^t \xrightarrow{A} \mathbb{Z}^s \rightarrow V \rightarrow 0$$

where  $A$  is the  $s \times t$  matrix of relations. We know that  $A$  can be diagonalized to the form such that it has  $a_1, \dots, a_t$  on the diagonal and the last  $s - t$  rows of  $A$  are zero. The  $a_i$  can be determined in the following way: As before let  $I_m \subset \mathbb{Z}$  be the ideal generated by the  $m \times m$  minor determinants of  $A$ , for  $1 \leq m \leq t$ . Then the fact that  $\mathbb{Z}$  is a PID and that  $I_m \supset I_{m+1}$  imply that  $I_m = (a_1a_2 \cdots a_m)$  for some integers  $a_i$ . Moreover, let  $1 \leq \tau \leq t$  be the smallest integer such that  $I_\tau \neq 0$  and  $I_m = 0$  for all  $m > \tau$  (We assume  $A \neq 0$  because in that case  $V \simeq R^s$  satisfies the requirement of the problem) We can choose  $a_m = 0$  for all  $m > \tau$ . Thus  $V = \mathbb{Z}/(a_1) \oplus \cdots \oplus \mathbb{Z}/(a_\tau) \oplus \mathbb{Z}^{s-t+\tau}$ , whence  $\text{rank}(V) = s - \tau \geq s - t$ .

**Proof 2** We prove a more general claim: Let  $R$  be an integral domain, and  $V$  be a finitely presented  $R$ -module with  $s$  generators and  $t$  relations where  $s \geq t$ , then  $\text{rank}(V) \geq s - t$ .

We have the exact sequence

$$R^t \xrightarrow{A} R^s \rightarrow V \rightarrow 0$$

where  $A$  is the  $s \times t$  matrix of relations. From the previous homework, for a multiplicative set  $S \subset R$ , the  $S^{-1}$  functor is exact and hence applying this to  $S = R - \{0\}$  and  $S^{-1}R = F$  (the fraction field of  $R$ ), we get the following exact sequence of  $F$ -vector spaces:

$$F^t \xrightarrow{A} F^s \rightarrow S^{-1}V \rightarrow 0$$

Since, for an  $s \times t$  matrix  $A$  over a field,  $\text{rank}(A) \leq \min(s, t) = t$ , we get  $\dim(S^{-1}V) \geq s - t$ . The fact that  $\text{rank}(V) = \dim_F(S^{-1}V)$  implies  $\text{rank}(V) \geq s - t$ .