Solutions to Homework 13

Math 600, Fall 2007

66 (10 points) Let F be any field, and $E \supset F$ any extension field. Let $A, B \in M_n(F)$. Suppose that A and B are similar when viewed as elements of $M_n(E)$. Show that A and B are similar as elements of $M_n(F)$.

The RCFs of A and B over F are the RCFs of A and B over E by uniqueness of RCF. Also the RCFs of two matrices are identical iff they are similar matrices. Since A and B are similar over E, they have the same RCF over E as well as over F, and hence they must be similar over F.

67 (10 points) Let F be any field, and $A, B \in M_n(F)$. Show that A and B are similar iff the matrices $XI - A$ and $XI - B$ have the same Smith forms in $M_n(F[X])$.

Let V and W be the $F[X]$ module structures on F^n induced by A and B respectively. Then $V \simeq W \Leftrightarrow A$ and B are similar $\Leftrightarrow$ A and B have the same RCF. (This is easy to show, for a proof see Theorem 15 on Page 476 of [D-F]). On the other hand, by the theory of RCF, we know that V has the presentation given by the following exact sequence (and a similar one for W):

$$0 \to F[X]^n \xrightarrow{XI-A} F[X]^n \xrightarrow{\phi} V \to 0$$

where ϕ takes the standard basis of $F[X]^n$ to the standard basis of $F^n = V$ and we extend this $F[X]$-linearly. From this presentation we know that the Smith form of $XI - A$ has the invariant factors of V on the diagonal. Thus $XI - A$ and $XI - B$ have the same Smith forms in $M_n(F[X]) \Leftrightarrow V \simeq W$.

68 (10 points) Prove that any matrix $A \in M_n(F)$ is similar to its transpose $A^t \in M_n(F)$.

There are $P, Q \in GL(n, F[X])$ such that $Q(XI - A)P^{-1} = P^{-t}(XI - A^t)Q^t$ is diagonal (the Smith form). Thus $XI - A$ and $XI - A^t$ have the same smith form, whence A and A^t are similar (by Problem 67).

Another Proof: It is easy to show that a Jordan block and hence a matrix in Jordan form is similar to its transpose, whence A is similar to A^t over the algebraic closure of F and hence over F by Problem 66.

69 (a) (5 points) Let $A \in M_n(F)$. Let $\bar{F}$ be an algebraic closure of F (you may assume this exists). Show that the minimal polynomial $\min(A)$ and the characteristic polynomial $\text{char}(A)$ have the same roots in $\bar{F}$ (neglecting multiplicities).

Let $a_1(X), \cdots a_m(X) \in F[X]$ be the invariant factors of the $F[X]$ module structure on F^n induced by A. Since $a_m(X) = \min(A)$ and $a_1(X)a_2(X)\cdots a_m(X) = \text{char}(A)$ and $a_i(X) \,|\, a_m(X)$, clearly each root of $\text{char}(A)$ over $\bar{F}$ is a root of $\min(A)$, and each root of $\min(A)$ is a root of $\text{char}(A)$.

69 (b) (5 points) Suppose that A is a 2×2 or a 3×3 matrix over a field F. Show that the invariant factors of A (hence the rational canonical form of A) can be computed once one knows $\text{char}(A)$ and $\min(A)$.

Note : degree of $\text{char}(A)$ is always n, it is incorrect to entertain the possibility $\deg(\text{char}(A)) < n$.

Let m denote the number of (non unit) invariant factors of the $F[X]$ module F^n induced by A (where $n = 2$ or 3). If $m = 1$, then $\text{char}(A)$ is the only invariant factor and RCF(A) is the companion matrix corresponding to $\text{char}(A)$. If $m = n$, then each invariant is $X - \lambda$ for some $\lambda \in F$ and $\text{RCF}(A) = \lambda I_n$. In the 3×3 case, $m = 2$ is the remaining possibility, with the invariants being $X - \lambda_1$ and $(X - \lambda_1)(X - \lambda_2)$ and $\text{RCF}(A)$ consists of a 1×1 block λ_1 and a 2×2 companion matrix corresponding to $(X - \lambda_1)(X - \lambda_2)$.

70 (10 points) We say $A \in M_n(F)$ is diagonalizable over F if there exists $P \in GL_n(F)$ such that PAP^{-1} is a diagonal matrix.
(a) Let $\bar{F}$ denote an algebraic closure of F. Show that $A \in M_n(F)$ is diagonalizable over $\bar{F}$ if and only if its Jordan form in $M_n(F)$ is diagonal.

(b) Let $A \in M_n(F)$. Show that A is diagonalizable over F if and only $\min(A)$ factors in $F[X]$ as a product of pairwise distinct linear factors.

a) Let A be diagonalizable over $\bar{F}$, then A is similar to a a diagonal matrix D over $\bar{F}$, which is its own Jordan form, and hence the Jordan form of A (because similar matrices have same JCF). Conversely, if the Jordan form of A is diagonal, then the fact that A is similar to its Jordan form, shows that A is diagonalizable.

b) The multiplicity of $X - \lambda$ in $\min(A)$ (over $\bar{F}$) is the size of the largest Jordan block corresponding to the eigenvalue λ. Thus, the Jordan form of A is diagonal iff $\min(A)$ has distinct roots over $\bar{F}$. Thus, if $\min(A)$ has $\deg(\min(A))$ number of distinct roots in F, then A is similar (over $\bar{F}$) to its JCF which is a diagonal matrix in $M_n(F)$. By Problem 66, A is diagonalizable over F. Conversely, if A is diagonalizable over F to a diagonal matrix D, then $D = \mathrm{JCF}(A)$ and hence the multiplicity of each of the $\deg(\min(A))$ roots of $\min(A)$ (which exist over F) is one.

71 (10 points) Use the "minors algorithm" discussed in class to find the invariant factors and the RCF for the matrix

$$A = \begin{pmatrix} -1 & -2 & 6 \\ -1 & 0 & 3 \\ -1 & -1 & 4 \end{pmatrix}$$

If $a_1(X), a_2(X)$ and $a_3(X)$ are the (possibly unit) invariant factors of the $F[X]$-module F^3 induced by A, then $(a_1(X) \cdots a_m(X))$ is the ideal in $F[X]$ generated by the $m \times m$ minor determinants of $XI - A$. Clearly $a_1(X) = 1$ because X and $X + 1$ are both in $(a_1(X))$. Also, $a_2(X)a_3(X) = \mathrm{char}(A) = (X - 1)^3$. Next, we note that all 2×2 minors of $I - A$ have zero determinant whence $X - 1 \mid a_2(X)$, and $X - 1$ is itself a 2×2 minor determinant, thus $X - 1 = a_2(X)$. Thus the invariant factors are $X - 1$ and $(X - 1)^2$ and the RCF of A is:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 2 \end{pmatrix}$$

Remark: In this problem it turns out that A is similar to its Jordan form even over $\mathbb{Z}$. For example, let $P = \begin{pmatrix} 3 & 2 & -1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$, then $P \in GL_3(\mathbb{Z})$ and $P^{-1}AP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$