Homework 14 – due 12/5/07

Math 600

72. In the following all exact sequences are in the category of R-modules.

(a) Let $M' \to M \to M'' \to 0$ be a sequence of R-modules and homomorphisms. Prove that this sequence is exact if and only if, for every R-module N the induced sequence

$$0 \to \mathrm{Hom}_R(M'', N) \to \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(M', N)$$

is exact.

(b) Let $0 \to N' \to N \to N''$ be a sequence of R-modules and homomorphisms. Prove that this sequence is exact if and only if, for every R-module M the induced sequence

$$0 \to \mathrm{Hom}_R(M, N') \to \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(M, N'')$$

is exact.

73. (5 points) Suppose we have a commutative diagram in $\underline{R-\mathrm{Mod}}$:

$$\begin{array}{ccccc} A' & \longrightarrow & A & \longrightarrow & A'' \\ \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ B' & \longrightarrow & B & \longrightarrow & B'' \end{array}$$

such that the vertical arrows are isomorphisms, and the first row is exact. Prove that the second row is exact.

74. Let R and S be rings. Consider functors $F : \underline{R-\mathrm{Mod}} \to \underline{S-\mathrm{Mod}}$ and $G : \underline{S-\mathrm{Mod}} \to \underline{R-\mathrm{Mod}}$. We say that F **is a left adjoint of** G (or that G **is a right adjoint of** F) provided that we have *natural* isomorphisms

$$\mathrm{Hom}_S(FX, Y) \xrightarrow{\sim} \mathrm{Hom}_R(X, GY)$$

for X an R-module and for Y an S-module. What does *natural* mean? By definition, it means that given $X' \to X$ in R-Mod and $Y \to Y'$ in S-Mod, the following diagram (with the arrows having the obvious meanings) is commutative:

$$\begin{array}{ccc} \mathrm{Hom}_S(FX, Y) & \xrightarrow{\sim} & \mathrm{Hom}_R(X, GY) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_S(FX', Y') & \xrightarrow{\sim} & \mathrm{Hom}_R(X', GY'). \end{array}$$

Prove the following statement: If F is left adjoint to G, then F is right exact and G is left exact.

75. Let R and S be commutative rings. Let M be an R-module, P an S-module, and N an (R,S)-bimodule (that is, simultaneously an R-module and an S-module and the two structures are compatible in the sense that $r(xs) = (rx)s$ for all $r \in R$, $s \in S$, $x \in N$). Prove that $M \otimes_R N$ is naturally an S-module, $N \otimes_S P$ an R-module, and that we have

$$(M \otimes_R N) \otimes_S P \cong M \otimes_R (N \otimes_S P).$$

76. Show that if m and n are coprime integers, then $\mathbb{Z}/m\mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z} = 0$.