Solutions to Homework 14

Math 600, Fall 2007

72 (10 points) In the following all exact sequences are in the category of R-modules, (R is commutative)

(a) Let $M' \xrightarrow{i} M \xrightarrow{j} M'' \to 0$ be a sequence of R-modules and homomorphisms. Prove that this sequence is exact iff, for every R-module N the following induced sequence is exact.

$$0 \to \mathrm{Hom}_R(M'', N) \xrightarrow{j^*} \mathrm{Hom}_R(M, N) \xrightarrow{i^*} \mathrm{Hom}_R(M', N)$$

Suppose the first sequence is exact, then for any N, we need to show that j^* is injective and $\mathrm{im}(j^*) = \ker(i^*)$. We have, $j^*(\phi) = 0 \;\Leftrightarrow \phi(jm'') = 0$ for all $m'' \;\Leftrightarrow \phi = 0$ by surjectivity of j. Thus j^* is injective. Also, $i^*(\phi) = 0 \;\Leftrightarrow \phi(im') = 0$ for all $m' \;\Leftrightarrow \phi = j^*(\psi)$, where in the last equivalence, we define $\psi(m'') = \phi(j^{-1}m'')$ (It can be checked that ψ is R-linear). Conversely, suppose the Hom sequence is exact for all N. Let $N = M''$, then $i^* \circ j^* = 0$, implies $i^* j^* id = 0$, which in turn implies $j \circ i = 0$ or $\mathrm{im}(i) \subset \ker(j)$. Next, let $N = M/i(M')$ and let $\pi : M \to M/i(M')$ be the quotient map, then clearly $i^*\pi = 0$ whence $\pi = j^*\phi = \phi \circ j$ for some $\phi \in \mathrm{Hom}_R(M'', M/i(M'))$. This implies $\ker(j) \subset \ker(\pi) = i(M')$, whence $\mathrm{im}(i) \supset \ker(j)$. Thus $\mathrm{im}(i) = \ker(j)$. Next, let $N = M''/j(M)$ and let $\pi : M'' \to M''/j(M')$ be the quotient map, then clearly $j^*\pi = 0$ whence $\pi = 0$ by injectivity of j^*. But $\pi = 0$ is equivalent to the surjectivity of j. Thus $M' \xrightarrow{i} M \xrightarrow{j} M'' \to 0$ is exact.

(b) Let $0 \to N' \xrightarrow{i} N \xrightarrow{j} N''$ be a sequence of R-modules and homomorphisms. Prove that this sequence is exact iff, for every R-module M the following induced sequence is exact.

$$0 \to \mathrm{Hom}_R(M, N') \xrightarrow{i'} \mathrm{Hom}_R(M, N) \xrightarrow{j'} \mathrm{Hom}_R(M, N'')$$

Suppose the first sequence is exact. To show that that i' is injective, let $i'\phi = 0$. This means $i \circ \phi = 0$ which implies $\phi = 0$ by injectivity of i. Next we show $\mathrm{im}(i') = \ker(j')$. Let $j'\phi = 0$, this means $j\phi(m) = 0$, so that there is a unique n' with $\phi(m) = i(n')$. Define $\psi(m) = n'$, then clearly $\psi = i'\phi$. Thus the Hom sequence is exact. Conversely, suppose the Hom sequence is exact for all M. Taking $M = N'$, we see $j'i'id = 0$ implies $ji = 0$ or $\mathrm{im}(i) \subset \ker(j)$. Taking $M = \ker(j)$, and $\phi : \ker(j) \to N$ to be the inclusion, we see $j'\phi = 0$ whence $\phi = i'\psi$ for some $\psi \in \mathrm{Hom}_R(\ker(j), N')$. Thus $k = i\psi(k)$ for all $k \in \ker(j)$, or $\ker(j) \subset \mathrm{im}(i)$. Thus $\ker(j) = \mathrm{im}(i)$. Next, taking $M =\ker(i)$, and $\phi : \ker(i) \to N'$ to be the inclusion, we see $i'\phi = 0$ whence $\phi = 0$ by the injectivity of i'. But

$\phi = 0$ is equivalent to the injectivity of i. Thus $0 \to N' \xrightarrow{i} N \xrightarrow{j} N''$ is exact.

73 (5 points) Suppose we have the following commutative diagram in $\underline{R\text{-Mod}}$, in which the vertical arrows are isomorphisms, and the first row is exact. Prove that the second row is exact.

$$\begin{tikzcd}
A' \arrow[r, "i"] \arrow[d, "f'"', "\cong"] & A \arrow[r, "j"] \arrow[d, "f"', "\cong"] & A'' \arrow[d, "f''"', "\cong"] \\
B' \arrow[r, "i'"] & B \arrow[r, "j'"] & B''
\end{tikzcd}$$

Solution:

$$j'(b) = 0 \Leftrightarrow j'f(f^{-1}b) = 0 \Leftrightarrow f''j(f^{-1}b) = 0 \Leftrightarrow j(f^{-1}b) = 0 \Leftrightarrow f^{-1}b = ia'$$
$$f^{-1}b = ia' \Leftrightarrow b = fia' \Leftrightarrow b = i'f'a' \Leftrightarrow b \in \operatorname{im}(i')$$

74 (10 points) Let R and S be (commutative) rings. Consider functors $F : \underline{R\text{-Mod}} \to \underline{S\text{-Mod}}$ and $G : \underline{S\text{-Mod}} \to \underline{R\text{-Mod}}$. We say that F is a left adjoint of G (or that G is a right adjoint of F) provided that we have *natural* isomorphisms:

$$\operatorname{Hom}_S(FX, Y) \xrightarrow{\sim} \operatorname{Hom}_R(X, GY)$$

for X an R-module and for Y an S-module. What does natural mean? By definition, it means that given $X' \to X$ in R-Mod and $Y \to Y'$ in S-Mod, the following diagram (with the arrows having the obvious meanings) is commutative:

$$\begin{tikzcd}
\operatorname{Hom}_S(FX, Y) \arrow[r, "\sim"] \arrow[d] & \operatorname{Hom}_R(X, GY) \arrow[d] \\
\operatorname{Hom}_S(FX', Y') \arrow[r, "\sim"] & \operatorname{Hom}_R(X', GY')
\end{tikzcd}$$

Prove the following statement: If F is left adjoint to G, then F is right exact and G is left exact.

Suppose F is left adjoint to G, we must show that given an exact sequence of R-modules and homomorphisms $X' \xrightarrow{i} X \xrightarrow{j} X'' \to 0$ the corresponding sequence of S-modules and homomorphism $FX' \xrightarrow{i} FX \xrightarrow{j} FX'' \to 0$ is exact. We know by Problem 72 a) that the following sequence is exact:

$$0 \to \operatorname{Hom}_R(X'', GY) \xrightarrow{j^*} \operatorname{Hom}_R(X, GY) \xrightarrow{i^*} \operatorname{Hom}_R(X', GY)$$

Further, the naturality of the isomorphism $\mathrm{Hom}_S(FX, Y) \xrightarrow{\sim} \mathrm{Hom}_R(X, GY)$, applied to the previous exact sequence yields the following commutative diagram:

$$\begin{tikzcd}
0 = \mathrm{Hom}_R(0, GY) \arrow[r] \arrow[d, "\sim"] & \mathrm{Hom}_R(X'', GY) \arrow[r] \arrow[d, "\sim"] & \mathrm{Hom}_R(X, GY) \arrow[r] \arrow[d, "\sim"] & \mathrm{Hom}_R(X', GY) \arrow[d, "\sim"] \\
0 = \mathrm{Hom}_S(F0, Y) \arrow[r] & \mathrm{Hom}_S(FX'', Y) \arrow[r] & \mathrm{Hom}_S(FX, Y) \arrow[r] & \mathrm{Hom}_S(FX', Y)
\end{tikzcd}$$

Although the first row is in the category R-Mod and the second row is in the category S-Mod, the whole diagram can be considered to be in the category $\mathbb{Z}$ -Mod. The result of Problem 73 applied twice to this diagram of $\mathbb{Z}$-modules, implies that the sequence in the second row of the above diagram is exact in $\mathbb{Z}$-Mod and hence in S-Mod. Applying the result of Problem 72 a) once again implies that the sequence $FX' \xrightarrow{i} FX \xrightarrow{j} FX'' \to 0$ is exact in S-Mod as desired. The proof that the functor G is left exact is similar.

75 (10 points) Let R and S be commutative rings. Let M be an R-module, P an S-module, and N an (R, S)-bimodule (that is, simultaneously an R-module and an S-module and the two structures are compatible in the sense that $r(xs) = (rx)s$ for all $r \in R,\ s \in S,\ x \in N$). Prove that $M \otimes_R N$ is naturally an S-module, $N \otimes_S$ P an R-module, and that we have:

$$(M \otimes_R N) \otimes_S P \cong M \otimes_R (N \otimes_S P)$$

Let A be the free abelian group on $M \times N$ and B be the subgroup such that $A/B = M \otimes_R N$. Define an S-module structure on V by $(m, n) \cdot s = (m, ns)$ and extend $\mathbb{Z}$-linearly. It is easy to check that B is an S- submodule of A and hence $A/B = M \otimes_R N$ inherits the S- module structure. It is also clear that $(m \otimes n)s = m \otimes ns$. Similarly $N \otimes_S P$ is an R-module.

Let $V = \mathcal{Z}(M \times N \times P)$ be the free abelian group on $M \times N \times P$. And let $V' = \mathcal{Z}(M \times (N \otimes_s P))$. Let $\Phi : V \to V'$ be defined by $\Phi((m, n, p)) = (m, (n \otimes p))$ and extend $\mathbb{Z}$-linearly. Clearly Φ is surjective. Let $W' \subset V'$ be the subgroup such that $V'/W' = M \otimes_R (N \otimes_S P)$. Let $W \subset V$ be the subgroup generated by relations of R-linearity in the first two factors of $M \times N \times P$ and the relations of S-bilinearity in the second two factors. It is easy to see that $\Phi(W) \subset W'$, whence we have a surjective map from $\bar{\Phi} : V/W \to V'/W'$. On the other hand it can be checked that $W' \subset \Phi(W)$: for example $(rm, n \otimes p) - (m, rn \otimes p) = \Phi((rm, n, p) - (m, rn, p))$. Thus $\bar{\Phi} : V/W \to M \otimes_R (N \otimes_S P)$ is an isomorphism of groups as well as (R, S)-bimodules. The definition of $W \subset V$ is symmetric in the sense that we have a similar isomorphism $\bar{\Psi} : V/W \to (M \otimes_R N) \otimes_S P$ of (R, S)-bimodules.

Thus $(M \otimes_R N) \otimes_S P \cong M \otimes_R (N \otimes_S P)$ as (R, S)-bimodules. Moreover it is also clear from the definition of Φ that the isomorphism between these two tensor product modules takes $m \otimes n \otimes p$ to itself. (Remark: Another proof, which is simpler, is given on Pg 371 of [D-F] under Theorem 14)

76 (10 points) Show that if m and n are coprime integers, then $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} = 0$

Clearly any element of $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$ is of the form $p1 \otimes 1$ for some $p \in \mathbb{Z}$. Let $a, b \in \mathbb{Z}$ such that $am + bn = 1$, then $1 \otimes 1 = (am + bn)1 \otimes 1 = a(m1 \otimes 1) + b(1 \otimes n1) = 0$. Thus $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} = 0$