Homework 15 – due 12/12/07

Math 600

77. (a) What is $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$?

(b) Show that $\mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} = 0$.

78. Let R be a PID, and $V \neq 0$ a f.g. R-module. Show that $V \otimes_R V \neq 0$. More precisely, find the structure of $V \otimes_R V$ (its rank, elementary invariants, and invariants).

79. Show that there is an isomorphism of $S^{-1}R$-modules

$$S^{-1}(M \otimes_R N) \quad \cong \quad S^{-1}M \otimes_{S^{-1}R} S^{-1}N,$$

which is natural in the R-modules M and N.

80. (a) Let R be a ring and I an ideal contained in the *Jacobson radical* of R (the intersection of all the maximal ideals in R). Let M be a f.g. R-module such that $M = IM$. Show that $M = 0$. (HINT: show $R/I \otimes M = 0$ and then use the criterion for the triviality of such tensor products which we proved in class. Alternatively, use a variant on our in-class proof of Nakayama's lemma.)

(b) Let R, I be as above. Let M be a f.g. R-module, and let $N \subset M$ be a submodule. Suppose $N + IM = M$. Show that $M = N$.

81. If $A \to B$ is a ring homomorphism and M is a flat A-module, show that $M_B := B \otimes_A M$ is a flat B-module. (The B module structure on the latter is given by $b(b' \otimes m) := bb' \otimes m$).

82. Let A,B,C, and D denote pairwise commuting $n \times n$ matrices over a field F. Show that

$$\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det(AD - BC).$$