

Solutions to Homework 15

Math 600, Fall 2007

77 (10 points) (a) What is $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$? (b) Show that $\mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} = 0$.

Let $i : \mathbb{Q} \rightarrow \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} : q \mapsto 1 \otimes q$. The map from $\mathbb{Q} \times \mathbb{Q} \rightarrow \mathbb{Q} : (q, q') \mapsto qq'$ is \mathbb{Z} -bilinear and hence induces $j : \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow \mathbb{Q}$. It is easy to show that $q \otimes q' = 1 \otimes qq'$ in $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$. Thus we see that $i \circ j = id$ and $j \circ i = id$, and $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}$.

We observe that $(a/b + \mathbb{Z}) \otimes (c/d + \mathbb{Z}) = (a/b + \mathbb{Z}) \otimes (cb/db + \mathbb{Z}) = (a + \mathbb{Z}) \otimes (c/db + \mathbb{Z}) = 0 \otimes (c/db + \mathbb{Z}) = 0$ whence $\mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} = 0$.

78 (10 points) Let R be a PID, and $V \neq 0$ a f.g. R -module. Show that $V \otimes_R V \neq 0$. More precisely, find the structure of $V \otimes_R V$ (its rank, elementary invariants).

Let $V = R^r \oplus_{i=1}^m R/(a_i)$ with $a_1 | a_2 | \cdots | a_m$. Then:

$$V \otimes_R V \simeq R^r \otimes R^r \bigoplus_{i=1}^m [R^r \otimes R/(a_i)] \bigoplus_{i=1}^m [R/(a_i) \otimes R^r] \bigoplus_{i,j=1}^m [R/(a_i) \otimes R/(a_j)]$$

Using the facts that tensor product distributes over direct sum, and $R/(a_i) \otimes R/(a_j) = R/(a_i, a_j)$, and $X \otimes_r A = X$ for any module X over a commutative ring A , the above expression simplifies to:

$$\begin{aligned} V \otimes_R V \simeq & R^{r^2} \bigoplus_{i=1}^m [R/(a_i)]^{2r} \bigoplus [R/(a_1)]^m \bigoplus [R/(a_1) \oplus (R/(a_2))^{m-1}] \\ & \bigoplus [R/(a_1) \oplus R/(a_2) + (R/(a_3))^{m-2}] \bigoplus \cdots \bigoplus [R/(a_1) \oplus \cdots R/(a_m)] \end{aligned}$$

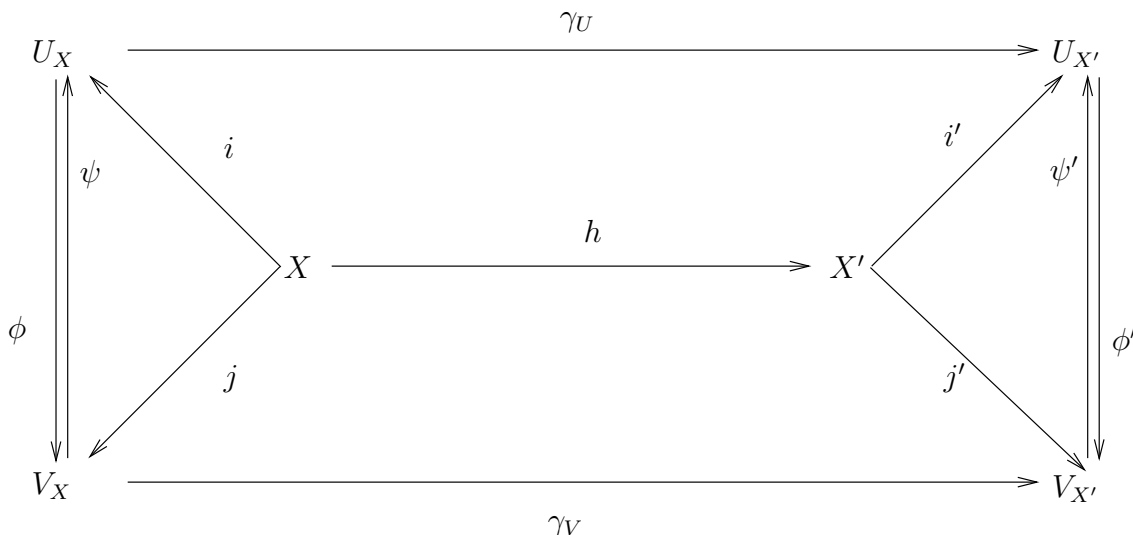
which further simplifies to :

$$V \otimes_R V \simeq R^{r^2} \oplus_{i=1}^m (R/(a_i))^{2(m-i+r)+1}$$

Thus the $\text{rank}(V \otimes_R V) = \text{rank}(V)^2$ (and not $2 \text{rank}(V)$). The invariant factors are the same a_i but with multiplicity $2(m - i + r) + 1$.

79 (10 points) Show that there is an isomorphism of $S^{-1}R$ -modules $S^{-1}(M \otimes_R N) \cong S^{-1}M \otimes_{S^{-1}R} S^{-1}N$ which is natural in the R -modules M and N .

Let X be an R -module, U_X an $S^{-1}R$ -module and $i : X \rightarrow U_X$ an R -module homomorphism, such that for any R -module homomorphism $\alpha : X \rightarrow Y$ from X to an $S^{-1}R$ -module Y , there is a *unique* $S^{-1}R$ -module homomorphism $\gamma : U_X \rightarrow Y$ such that $\gamma \circ i = \alpha$. If $j : X \rightarrow V_X$ also satisfies this universal property, then it follows that there are maps $\phi : U_X \rightarrow V_X$ and $\psi : V_X \rightarrow U_X$ such that $\phi \circ i = j$ and $\psi \circ j = i$ and $\phi \circ \psi = id$ and $\psi \circ \phi = id$ (show !). We want to show that the isomorphism between any two solutions of this universal mapping problem is natural in the R -module X . More precisely: Let $h : X \rightarrow X'$ be a given R -module homomorphism, and as above let $i' : X' \rightarrow U_{X'}$ and $j' : X' \rightarrow V_{X'}$ be any two solutions of the above universal mapping problem for X' . Let ϕ' and ψ' be the isomorphisms between $U_{X'}$ and $V_{X'}$ as above. The map $i' \circ h : X \rightarrow U_{X'}$ induces a unique map $\gamma_U : U_X \rightarrow U_{X'}$ satisfying $\gamma_U \circ i = i' \circ h$. Similarly the map $j' \circ h : X \rightarrow V_{X'}$ induces a unique map $\gamma_V : V_X \rightarrow V_{X'}$ satisfying $\gamma_V \circ j = j' \circ h$. We want to show that $\gamma_V \circ \phi = \phi' \circ \gamma_U$. Now observe that $(\phi' \circ \gamma_U \circ \phi^{-1}) \circ j = \phi' \circ \gamma_U \circ \psi \circ j = \phi' \circ \gamma_U \circ i = \phi' \circ i' \circ h = j' \circ h$ whence by the uniqueness of γ_V we must have $\phi' \circ \gamma_U \circ \phi^{-1} = \gamma_V$ or $\phi' \circ \gamma_U = \gamma_V \circ \phi$ as required. Given



R -modules M, N, M', N' with $f : M \rightarrow M'$ and $g : N \rightarrow N'$, let $X = M \otimes N$, $X' = M' \otimes N'$ and $h = f \otimes g$. By construction $S^{-1}(M \otimes_R N) = U_X$ and $S^{-1}(M' \otimes_R N') = U_{X'}$. We will prove that we can take $V_X = S^{-1}M \otimes_{S^{-1}R} S^{-1}N$ and $V_{X'} = S^{-1}M' \otimes_{S^{-1}R} S^{-1}N'$. Once we prove this, $\gamma_V \circ \phi = \phi' \circ \gamma_U$ implies that the isomorphism between $S^{-1}(M \otimes_R N)$ and $S^{-1}M \otimes_{S^{-1}R} S^{-1}N$ is natural in M and N . Given $\alpha : M \otimes N \rightarrow Y$ where Y is any $S^{-1}R$ -module, we observe that

we have an R -bilinear map $M \times N \rightarrow S^{-1}M \otimes_{S^{-1}R} S^{-1}N : (m, n) \mapsto m \otimes n$ and hence an R -module homomorphism $j : M \otimes N \rightarrow S^{-1}M \otimes_{S^{-1}R} S^{-1}N$. The map from $S^{-1}M \times S^{-1}N \rightarrow Y$ given by $(m/s, n/t) \mapsto f(m, n)/st$ is firstly independent of the equivalence classes of m/s and n/t in $S^{-1}M$ and $S^{-1}N$ respectively (check), and secondly it is $S^{-1}R$ -bilinear, hence we get a map $\gamma : S^{-1}M \otimes_{S^{-1}R} S^{-1}N \rightarrow Y$. Since $\gamma \circ j = f$ we have shown that $V_X = S^{-1}M \otimes_{S^{-1}R} S^{-1}N$ and $V_{X'} = S^{-1}M' \otimes_{S^{-1}R} S^{-1}N'$ are valid choices.

80 (10 points) (a) Let R be a ring and I an ideal contained in the Jacobson radical of R (the intersection of all the maximal ideals in R). Let M be a f.g. R -module such that $M = IM$. Show that $M = 0$.

From the lemma that $R/I \otimes M = M/IM$ we see that $R/I \otimes M = 0$. Note that both R/I and M are f.g. modules and we use the fact that for f.g. modules M, N , $\text{Ann}(M \otimes N) = \text{Ann}(M) + \text{Ann}(N)$. Therefore $R = I + \text{Ann}(M)$. Since $1 + I \subset R^\times$, it follows that $\text{Ann}(M) = R$, or $M = 0$.

(b) Let R, I be as above. Let M be a f.g. R -module, and let $N \subset M$ be a submodule. Suppose $N + IM = M$. Show that $M = N$.

Consider the R -module quotient homomorphism $M \rightarrow M/N$, we get $I(M/N) = (N + IM)/N$. Thus $(M/N)/I(M/N)$ is isomorphic to $(M/N)/(N + IM/N) \simeq M/(N + IM)$. Since we are given $M = N + IM$ we conclude that $I(M/N) = M/N$, whence by part a) and the fact that M/N is f.g., it follows that $M/N = 0$.

81 (10 points) If $f : A \rightarrow B$ is a ring homomorphism and M is a flat A -module, show that $M_B := B \otimes_A M$ is a flat B -module. (The B -module structure on the latter is given by $b(b' \otimes m) := bb' \otimes m$).

Let $0 \rightarrow X \xrightarrow{g} Y$ be an exact sequence of B -modules, then we have an induced map $X \otimes_B (B \otimes_A M) \xrightarrow{g \otimes id \otimes id} Y \otimes_B (B \otimes_A M)$ which we must show to be injective, in order to establish that M_B is flat. By Problem 75 (hw14), we know that $X \otimes_B (B \otimes_A M) \cong (X \otimes_B B) \otimes_A M$. Since $0 \rightarrow X \rightarrow Y$ is exact and B is flat over itself, we see that $0 \rightarrow X \otimes_B B \xrightarrow{g \otimes id} Y \otimes_B B$ is exact in $A\text{-Mod}$ and hence the flatness of M implies that $0 \rightarrow (X \otimes_B B) \otimes_A M \xrightarrow{g \otimes id \otimes id} (Y \otimes_B B) \otimes_A M$ is exact in the category of (A, B) -bimodules. The first row of the following commutative diagram is exact and the

vertical arrows are isomorphisms and hence the second row is exact in (A, B) -bimod as required.

$$\begin{array}{ccccc}
 0 & \longrightarrow & (X \otimes_B B) \otimes_A M & \xrightarrow{g \otimes id \otimes id} & (Y \otimes_B B) \otimes_A M \\
 \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\
 0 & \longrightarrow & X \otimes_B (B \otimes_A M) & \xrightarrow{g \otimes id \otimes id} & Y \otimes_B (B \otimes_A M)
 \end{array}$$

82 (10 points) Let A, B, C , and D denote pairwise commuting $n \times n$ matrices over a field F .

Let $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, show that $\det(M) = \det(AD - BC)$.

Let $D_t = D - tI$ then the matrices A, B, C , and D_t are pairwise commuting in $M_n(F(t))$. We have

$$\begin{bmatrix} A & B \\ C & D_t \end{bmatrix} \begin{bmatrix} D_t & 0 \\ -C & I \end{bmatrix} = \begin{bmatrix} AD_t - BC & B \\ 0 & D_t \end{bmatrix}$$

whence $\det(M_t) \det(D_t) = \det(D_t) \det(AD_t - BC)$ where M_t is M with D replaced by D_t . Since $\det(D_t) \neq 0$ in $F(t)$, we obtain that $\det(M_t) - \det(AD_t - BC) = 0$ which is an equation in $F[t]$. Therefore $\det(M) - \det(AD - BC)$ being the image of $\det(M_t) - \det(AD_t - BC)$ under the evaluation map $F[t] \xrightarrow{t \rightarrow 0} F$ is itself zero.