Solutions to Homework 1: MATH600, Fall 2007

1. (5 pts) Let S be any set with more than one element and define binary operation $s_1 s_2 = s_2$. This product is associative: $(s_1 s_2)s_3 = s_1(s_2 s_3) = s_3$. Any $e \in S$ satisfies $es = s$ and $se = e$ $\forall s \in S$. From these two identities we infer:

i) e is a left identity

ii) every element has a right inverse with respect to e

iii) no such e is a right identity when S has at least two elements.

Thus S satisfies the requirement of the problem but fails to be a group via failure of existence of a 'two-sided' identity element.

2. (a) (5 pts) Let $r \in \mathbb{R}$. We may assume $\phi(r) \neq -1$ by replacing r with $-r$ if needed. We have $\phi(\frac{r}{\phi(r)+1}) = \frac{\phi(r)}{\phi(r)+1}$ whence $\phi(r)+1$ divides 1, or equivalently $\phi(r) = 0$. Thus $\phi \equiv 0$

(b) (5 pts) Letting $r \in \mathbb{Q}$ in the proof of $2(a)$ above, we see $\phi(r) = 0$. Thus any group hmm. $\phi : \mathbb{Q} \to \mathbb{Z}$ is trivial, whence $\mathbb{Q}$ and $\mathbb{Z}$ are not isomorphic groups.

(c) (5 pts) Let $\phi : \mathbb{R}^\times \to \mathbb{R}_{>0}$ be a homomorphism of groups. The relation $\phi(-1)^2 = \phi(1) = 1$ implies that $\phi(-1) = 1$ whence $\ker(\phi)$ is non trivial and ϕ cannot be an isomorphism.

3. (a) (5 pts) The fact that $f_{|\mathbb{Q}} = id$ follows from $f(1) = 1$. Continuity of f forces $f_{|\mathbb{R}} = id$. If f is not continuous it is possible to construct a nontrivial isomorphism of $\mathbb{Q}$-vector spaces $f : \mathbb{R} \to \mathbb{R}$ satisfying $f(1) = 1$ as below. In particular f will be a non-trivial automorphism of $(\mathbb{R}, +)$

Let B be a basis for $\mathbb{R}$ over $\mathbb{Q}$ (every vector space has a basis using Zorn's lemma). Then there is a unique rational number $b_0 \in B$. Let σ be a nontrivial permutation of B that fixes b_0. Define $f_{|B} = \sigma$ and extend over the scalars $\mathbb{Q}$ to define f. It is easily verified that f is a vector space isomorphism.

3. (b) (5 pts) By $f(1) = f(1)^2$, it follows that $f(1) = 0$ or 1. In the former case $f \equiv 0$ since we have $\forall r \in \mathbb{R}$, $f(r) = f(r)f(1) = 0$. Now, supposing $f(1) = 1$ we easily deduce as before that $f_{|\mathbb{Q}} = id$. Now we make a (tricky) observation that $x > 0 \Leftrightarrow f(x) > 0$. To see this, for positive x we have $\phi(x) = \phi(\sqrt{x})^2 > 0$ and for $x < 0$ we have $-\phi(x) = \phi(-x) > 0$. Thus f is strictly increasing. [Aside: f non-decreasing suffices for the argument but you may want to prove that the additive and multiplicative property of f force it to be either zero or one-to-one]. Now suppose $x \neq f(x)$, we may assume $x < f(x)$ by replacing x with $-x$ if necessary. Pick a rational number y with $x < y < f(x)$ and apply f to the left inequality to infer $f(x) < f(y) = y$ which contradicts the right inequality. This contradiction shows that $f(x) = x \ \forall x \in \mathbb{R}$