

Homework 2 – due 09/12/07

Math 600

4. (5 pts) Show that a group is never the union of two proper subgroups.
5. (10 points) Dummit-Foote, 2.3, #16.
6. (10 points) Dummit-Foote, 2.3, #21-23.
7. (10 points) Dummit-Foote, 2.3, #25-26.
8. (a) (5 points) Let A and B be subgroups of a group G . Show that $AB = \{ab \mid a \in A, b \in B\}$ is a subgroup if and only if $AB = BA$.
(b) (5 points) If $|A|, |B| < \infty$, then $|AB| = \frac{|A||B|}{|A \cap B|}$.
9. (10 points total) Let H be a subgroup of a group G . Consider the left action of G on the coset space G/H . Let $\phi : G \rightarrow \Sigma(G/H)$ be the associated group homomorphism.
 - (a) Show $\ker(\phi) = \bigcap_{g \in G} gHg^{-1}$.
 - (b) Show that if $[G : H] = n$, then G has a normal subgroup whose index divides $n!$.
 - (c) Deduce that if G is simple (i.e. has no non-trivial normal subgroups), and $H \subset G$ is a subgroup of index r , then $|G|$ divides $r!$.
 - (d) Deduce that if G is a simple group of order 60, and H is a subgroup of G , then $|H| \leq 12$.
10. (10 points) Let G be a finite group which acts on a set X . Let p be a prime number. Suppose that for each $x \in X$, the stabilizer G_x contains a p -group P_x which fixes only x . Show that G acts transitively on X . HINT: Let $x, y \in X$ with $y \notin \mathcal{O}_x$. Think about P_x and P_y acting on \mathcal{O}_x . What can you say about $|\mathcal{O}_x|$ modulo p ?