Homework 4 – due 09/26/07

Math 600

15. (5 points) Suppose $n \geq 3$. Show that $Z(S_n) = 1$.

16. (10 points) Suppose G is a group with $|G| > 2$ (the group need not be finite). Show that $|\mathrm{Aut}(G)| > 1$. You may assume that every vector space has a basis.

17. Prove the following assertions (the first two we made in lecture).
(a) (5 points) The only elements in S_5 which commute with $\pi := (1\,2\,3\,4\,5)$ are even. In fact, show the centralizer of π is simply $\langle \pi \rangle$.

(b) (5 points) The elements $(1\,2\,3\,4\,5)$ and $(1\,3\,5\,2\,4)$ are conjugate in S_5 but not in A_5.

(c) (5 points) Show that $(1\,2\,3\,4\,5)$ and $(1\,3\,5\,2\,4)$ are conjugate as members of A_7 but not as members of A_6.

18. (10 points) Show that a d-cycle in S_n cannot be written as a product of fewer than $d-1$ transpositions. HINT: Say $\gamma = \sigma_1 \cdots \sigma_k$, where σ_i is a transposition. Consider $P(\gamma) = P(\sigma_1) \cdots P(\sigma_k)$ in $\mathrm{GL}_n(\mathbb{C})$. Let $H_i = \ker(P(\sigma_i) - 1)$ and $K = \ker(P(\gamma) - 1)$. Show that $K \supseteq \cap_i H_i$. On the other hand, show that $\dim(\cap_i H_i) \geq n - k$ and $\dim(K) = (n - d) + 1$.

19. (10 points) Let G be a finite group acting on a finite set X. For $g \in G$, let $f(g)$ denote the number of elements in X which are fixed by g. Prove Burnside's formula

$$\frac{1}{|G|} \sum_{g \in G} f(g) = \text{number of orbits.}$$