

Solutions to Homework 4

Math 600, Fall 2007

15 a) (5 pts) $Z(S_n) = 1$ for $n \geq 3$

If $g \in Z(S_n)$, then by normality, $Z(S_n)$ contains all g' which have the same cycle structure as g . We know that any such g' is conjugate to g , but $g \in Z(S_n)$ implies $g = g'$. Thus g is the unique element of S_n with the cycle structure of g , which can only happen if $n = 2$ and g is the nontrivial element of S_2 .

16 (10 pts) $|G| > 2 \Rightarrow |\text{Aut}(G)| > 1$

If G is nonabelian then G has $[G : Z(G)] > 1$ number of inner automorphisms. If G is abelian $g \mapsto g^{-1}$ is an automorphism which is nontrivial unless $g^2 = 1 \forall g \in G$. Examples of the latter case are $G = \prod_{\alpha \in \mathcal{A}} C_2$ where \mathcal{A} is a possibly uncountable index set, or the subgroup of G consisting of tuples with only finitely many nonzero entries. The second example shows that G need not be a direct product of C_2 's as groups. However, we see that G is always a \mathbb{F}_2 vector space, simply by using additive notation. Infact, the notions \mathbb{F}_2 -vector space and abelian group G with $2G = 0$ are identical. Since every vector space has a basis $\{x_\beta, \beta \in \mathcal{B}\}$, we have $G = \bigoplus_{\beta \in \mathcal{B}} \mathbb{F}_2 x_\beta$. Using the axiom of choice, there is a nontrivial permutation σ of \mathcal{B} . Consider $\phi : G \rightarrow G$ specified by $\phi(x_\beta) = x_{\sigma\beta}$ and extend linearly over \mathbb{F}_2 . Again ϕ being a vector space homomorphism means that it is a group homomorphism.

17a) (5 pts) Let $\pi = (12345)$ in S_5 , show $Z(\pi) = \langle \pi \rangle$

Let $\sigma \in Z(\pi)$ then we have

$$(12345) = \sigma(12345)\sigma^{-1} = (\sigma(1)\sigma(2)\sigma(3)\sigma(4)\sigma(5))$$

This implies $\sigma = \pi^i$ where $i = \sigma(1)$. Thus $Z(\pi) \subset \langle \pi \rangle$ which together with $Z(\pi) \supset \langle \pi \rangle$ (since $\langle \pi \rangle$ is abelian) implies $Z(\pi) = \langle \pi \rangle$

17b) (5 pts) $\pi = (12345)$ and $\tilde{\pi} = (13524)$ are conjugate in S_5 but not in A_5

We want to find $\sigma \in S_n$ such that $\tilde{\pi} = \sigma\pi\sigma^{-1}$, which is the same as

$$(13524) = (\sigma(1)\sigma(2)\sigma(3)\sigma(4)\sigma(5))$$

Therefore σ is determined once we pick $\sigma(1)$. We have five possibilities for $\sigma(1)$, namely 1, 3, 5, 2, 4 which are also $\tilde{\pi}^i(1)$ for $i = 0..4$ respectively. Let us call the σ resulting from the choice $\sigma(1) = \tilde{\pi}^i(1)$

by σ_i , $i = 0..4$. We have, $\sigma_0 = (2354)$ and for $i = 1..4$ we get $\sigma_i = \tilde{\pi}^i \sigma_0$. Note that $\tilde{\pi} \in A_5$ and $\sigma_0 \notin A_5$, therefore $\sigma_i \notin A_5 \forall i$.

17c) (5 pts) If we are working in S_7 instead of S_5 , then in the notation of part b), we have ten choices for σ namely σ_i and $\sigma_i(67)$. The latter five are in A_7 but not in A_6 .

18) (10 pts) Let S_n act on the set $\{x_1, x_2, \dots, x_n\}$ by $\sigma : x_i \mapsto x_{\sigma(i)}$. Let \mathfrak{K} be a field, for example \mathbb{F}_2 and let V be the \mathfrak{K} -vector space with basis $\{x_1, x_2, \dots, x_n\}$. We have $V = \bigoplus_{i=1}^n \mathfrak{K}x_i$. Extend the action of S_n on the basis linearly to get action of S_n on V . This action satisfies the property that $\sigma(k \cdot v) = k \cdot \sigma(v)$ where $k \in \mathfrak{K}$, $v \in V$. This means we have a homomorphism $\rho : S_n \rightarrow GL(n, \mathfrak{K})$. It is clear that the matrices in $\rho(S_n)$ have the property that each column contains exactly one entry with value 1 and other entries are zero. In fact $\rho(\sigma)_{ij} = 1$ iff $\sigma(j) = i$. Thus we see that $\rho(\sigma) = 1$ implies $\sigma = 1$. We can therefore identify S_n with $\rho(S_n)$.

Now let $\gamma = \sigma_1 \cdots \sigma_k$ be a d -cycle where the σ_i are transpositions. Let H_i be the subspace of V on which σ_i acts as identity. Then clearly $\gamma = \sigma_1 \cdots \sigma_k$ acts as identity on $\bigcap H_i$. In other words $K \supseteq \bigcap H_i$ where K is the subspace of V fixed by γ . If $\sigma_i = (12)$ then it is easily seen that $H_i = \mathfrak{K}(x_1+x_2) \oplus_{i=3}^n \mathfrak{K}x_i$ thus we see in general that $\dim(H_i) = n - 1$. Similarly $K = \mathfrak{K}(x_1 + \dots + x_d) \oplus_{i=d+1}^n \mathfrak{K}x_i$ (when $\gamma = (12 \cdots d)$) so that $\dim(K) = n - d + 1$. Define an inner product on V by $\langle x_i, x_j \rangle = \delta_{ij}$ (in other words $\langle v, w \rangle = v^t w$). Pick vectors z_i such that $H_i^\perp = \mathfrak{K}z_i$. Now $x \in \bigcap H_i \Leftrightarrow x \in (\bigoplus_{i=1}^k \mathfrak{K}z_i)^\perp$. Thus $\dim(\bigcap H_i) = n - \dim(\bigoplus_{i=1}^k \mathfrak{K}z_i)$ and clearly $k \geq \dim(\bigoplus_{i=1}^k \mathfrak{K}z_i)$, therefore $\dim(\bigcap H_i) \geq n - k$. Thus $n - d + 1 \geq n - k$ or $k \geq d - 1$.

Another direct proof of 18) (by Ryan Zavislak)

Proposition: Let $\sigma \in S_n$ and $\sigma = \sigma_1 \cdots \sigma_k$ where the σ_i are transpositions. Let $\sigma = c_1 c_2 \cdots c_l$ be the cycle decomposition of σ (including 1-cycles), then $l \geq n - k$

Corollary: If $\sigma = (1, 2, \dots, d) = \sigma_1 \cdots \sigma_k$ where the σ_i are transpositions, then $k \geq d - 1$

Proof of Corollary: In the notation of the proposition, we have $l = n - d + 1$ by definition of σ , and we have $l \geq n - k$ by the proposition, whence $k \geq d - 1$

Proof of Proposition: We will use the formulae:

$$(1, j)(1, 2, \dots, m) = (1, 2, \dots, j-1)(j, j+1, \dots, m)$$

$$(1, 2 \cdots j)(j, j+1)(j+1, j+2, \dots, m) = (1, 2, \dots, j-1, j+1, j+2, \dots, m)$$

The proposition is true for $k = 1$. Assume it is true for $k \leq m - 1$. Given $\sigma = \sigma_1 \cdots \sigma_m$, let l denote the number of cycles (including 1-cycles) in the decomposition of $\sigma_2 \cdots \sigma_m$. Consider the

product of σ_1 with the cycle decomposition of $\sigma_2 \cdots \sigma_m$, then we are (essentially) in the situation of the two formulae mentioned, whence the number of cycles in the decomposition of σ is $l \pm 1$. By the inductive step, we have $l \geq n - m + 1$. Therefore, $l \pm 1 \geq l - 1 \geq n - m$ as was to be shown.

19) (10 pts) Let n be the number of orbits. Let x_i , $i = 1..n$, be the representatives of the n orbits which we denote by O_{x_i} . By definition g fixes x iff $g \in G_x$. Also recall that $|G_{gx}| = |G_x|$ because $G_{gx} = g G_x g^{-1}$. Thus:

$$\sum_{g \in G} f(g) = \sum_{x \in X} |G_x| = \sum_{i=1}^n \sum_{x \in O_{x_i}} |G_{x_i}| = \sum_{i=1}^n |G| = n |G| \quad \text{QED}$$