Solutions to Homework 5

Math 600, Fall 2007

20 (5 pts) If $n \geq 5$ then A_n is the only proper normal subgroup of S_n
Let $N \triangleleft S_n$. Since A_n is simple $(n \geq 5)$, we have $N \cap A_n = A_n$ or trivial. In the former case we must have $N = A_n$ or S_n because $[S_n : A_n] = 2$. In the latter case, let x and y be (possibly identical) nontrivial elements of N, then $xy \in A_n \cap N$ whence $xy = 1$. This shows $N = \{1, x\}$ and hence x is the only element of S_n having the cycle structure of x, which is impossible if $n > 2$, hence N must be trivial

21 a) (5 pts) Let $N \triangleleft G$ be nontrivial p-groups then $N \cap Z(G) > 1$
Since N is normal, G acts on N by conjugation, and let n_s be the number of singleton orbits. We have $n_s = |Z(G) \cap N| \geq 1$. We now conclude $|Z(G) \cap N| > 1$ by using the fact that when a nontrivial p-group acts on a set with p^n elements $(n \geq 1)$, and if $n_s > 0$ then $n_s = p \cdot l$ for some $l \geq 1$

21 b) (5 pts) G nonabelian and $|G| = p^3 \Rightarrow |Z(G)| = p$ and $Z(G)$ has no complement in G
Since a p-group has nontrivial center and G is nonabelian, we have $|Z(G)| = p$ or p^2
We use the following fact: If $G/Z(G)$ is cyclic then $G = Z(G)$ (Prove !). In the present case, this tells us that $|Z(G)| = p$. Also, if H is a complement of $K = Z(G)$ then $G = H \cdot K \simeq H \times K$ (because here, $hk = kh$) shows that $H \triangleleft G$, but then $H \cap Z(G) = 1$ contradicts part a) of this problem, thus $Z(G)$ has no complement.

22 (10 pts) A group G of order $2^2 \cdot 5 \cdot 19$ is not simple.
Suppose, on the contrary that G is simple, then the number of sylow 5-groups, $n_5 = 76$ and similarly $n_{19} = 20$. (Using $n_5 | 76$ and $n_5 \equiv 1 \pmod{5}$ etc.). Also from the Sylow theorems, every element of order p is in some sylow p-group. Therefore the number of elements of order 5 and 19 are $76 \cdot 4$ and $20 \cdot 18$ which add up to more than $|G|$. This contradiction shows G cannot be simple.

23 (10 pts) A_n is generated by (123) and $(23\cdots n)$ if $n \geq 4$ is even
Let $a = (123)$ and $b = (23\cdots n)$. Let $H = \langle a, b\rangle < A_n$. We must show $H = A_n$. We will show this by proving that $H \lhd S_n$ and using the result of Problem 20) (above) that the only proper normal subgroup of S_n is A_n if $n \geq 5$. Observe that:

$$S_n = \langle\, (12), (23), \cdots, (n{-}1\, n)\,\rangle = \langle\, (12), (123\cdots n)\,\rangle = \langle\, (12), (12)b\,\rangle = \langle\, (12), b\,\rangle$$

Therefore $H \lhd S_n \Leftrightarrow (12)\, H\, (12) = H$. We have $(12)a(12) = a^{-1}$ and $(12)b(12) = a^{-1}b$, by direct calculation. But this implies $H \lhd S_n$ and completes the proof that $H = A_n$

24 (15 pts) $|G| = p^b n$ with $p|n$ allowed, $H < G$ with $|H| = p^a$ and $0 \leq a \leq b$

a) Let Ω be the set of subsets of G with p^b elements which are stable under left multiplication by H. In other words each $M \in \Omega$ is a union of p^{b-a} right cosets of H. Therefore G acts on Ω by $g : M \mapsto Mg^{-1}$. [Note: This is inessential, but we have to right multiply by g^{-1}, instead of g, so that we get an action of G, instead of the opposite group G^{opp}]. Show that the stabilizer G_i of an element $M_i \in \Omega$ satisfies $|G_i| \mid p^b$.
Indeed, G_i acts on M_i by right multiplication $g : m \mapsto mg^{-1}$ and clearly each orbit has $|G_i|$ elements (since multiplication by g is a bijective mapping), whence $|G_i|$ divides $|M_i| = p^b$.

b) Returning to the G-set Ω: the orbit of $M_i \in \Omega$ has n elements $\Leftrightarrow |G_i| = p^b$
Proof : By part a) $|G_i| = p^c$ for some $c \leq b$. Therefore:

$$|\text{Orbit}(M_i)| = [G : G_i] = p^{b-c} \cdot n$$

whence we obtain that $|\text{Orbit}(M_i)| \geq n$ with equality iff $|G_i| = p^b$

c) Consider an orbit $\mathcal{T} = \{M_1, M_2, \cdots, M_k\}$ with $k = p^{b-c} \cdot n$ elements (by part b). Let $m \in M_1$, the orbit $\mathcal{T}$ satisfies $\mathcal{T} = M_1 \cdot G$, therefore $\cup_{i=1..k} M_i \supset m \cdot G = G$. Since the M_i overlap in general, we have

$$p^b \cdot n = |G| = |\bigcup_{i=1..k} M_i| \leq \sum_{i=1..k} |M_i| = p^{b-c} \cdot n \cdot p^b$$

with equality if and only if: $c = b \;\Leftrightarrow\; \mathcal{T}$ has exactly n elements $\;\Leftrightarrow M_1 \,..\, M_k$ are pairwise disjoint. Consider such an orbit $\mathcal{T}$ with n elements $M_1 \,..\, M_n$: Since $\coprod_{i=1..n} M_i = G$, assume, wolog that $1 \in M_1$. If $g \in M_1$ then $g \in M_1 \cdot g$ therefore $M_1 = M_1 \cdot g$, in other words M_1 is a subgroup of order p^b containing H (since each M_i is a union of right cosets of H). Conversely if U is a subgroup of G with $H < U$ and $|U| = p^b$, then considering U as an element of the G-set Ω, we see the orbit of U has exactly n elements, which are just the right cosets of U in G.

d) Let $N(p^b, H)$ denote the number of subgroups U with $H < U$ and $|U| = p^b$. In other words $N(p^b, H) = \#\{\text{orbits}\mathcal{T} \ : \ |\mathcal{T}| = n\}$. We have:

$$|\Omega| = \sum_{|\mathcal{T}|=n} |\mathcal{T}| + \sum_{pn \,|\, |\mathcal{T}|} |\mathcal{T}| \equiv n \cdot N(p^b, H) \pmod{np}$$

e) Each element M of Ω is a (disjoint) union of p^{b-a} right cosets of H (as noted in part a)) and the right cosets are $p^{b-a} \cdot n$ in number, whence $|\Omega| = \binom{p^{b-a} \cdot n}{p^{b-a}}$, and combining this with the deduction of part d) we get:

$$\binom{p^{b-a} \cdot n}{p^{b-a}} \equiv n \cdot N(p^b, H) \pmod{np}$$

f) When G is cyclic, it has exactly one subgroup of any order (which divides $|G|$) therefore

$$\binom{p^{b-a} \cdot n}{p^{b-a}} \equiv n \pmod{np}$$

g) Observe that the LHS of the above equation is independent of the group G, whence

$$n \cdot N(p^b, H) \equiv n \pmod{np} \ \Rightarrow \ N(p^b, H) \equiv 1 \pmod{p}$$