

ON THE COHOMOLOGY OF SIMPLE SHIMURA VARIETIES WITH NON QUASI-SPLIT LOCAL GROUPS

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ABSTRACT. We study the Scholze test functions for bad reduction of simple Shimura varieties at a prime where the underlying local group is any inner form of a product of Weil restrictions of general linear groups. Using global methods, we prove that these test functions satisfy a vanishing property of their twisted orbital integrals, and we prove that the pseudostabilization base changes of such functions exist (even though the local group need not be quasi-split) and can be expressed in terms of explicit distributions in the stable Bernstein center. We then deduce applications to the stable trace formula and local Hasse-Weil zeta functions for these Shimura varieties.

CONTENTS

1.	Introduction	1
2.	Preparations in local harmonic analysis	5
3.	p -divisible groups with EL structure and test functions	22
4.	Simple Shimura varieties of Kottwitz type	26
5.	Proof of the local results	35
	References	47

1. INTRODUCTION

In this paper, we study the bad reduction of the simple Shimura varieties attached to unitary similitude groups introduced in the work of Kottwitz [Kot92a]. Our goal is twofold. On the one hand, we extend the results of [SS13] that require the local group to be a product of Weil restrictions of general linear groups, instead allowing the local group to be any inner form of such a group. We thus systematically deal with issues arising from the failure of the local group to be quasi-split. This extension includes the proof of a vanishing property of twisted orbital integrals of various test functions and a theory of pseudostabilization base change transfer for such functions (both of these are new features arising for non-quasi-split groups). Further, it includes a comparison of certain generalizations of the Scholze test functions defined in [Sch13b] with certain standard test distributions in the stable Bernstein center detailed in [Hai14]. On the other hand, we use these local results to describe the cohomology and the semisimple Hasse-Weil local factors of Kottwitz-type simple Shimura varieties, generalizing previous works, such as [SS13], [She18] and [She19], to more cases of bad reduction.

1.1. Main results. Let $\mathbb{F}$ be a CM field with totally real subfield $\mathbb{F}_0$. Let $(\mathbb{D}, *)$ be a division algebra with center $\mathbb{F}$ together with an involution $*$ of second kind, giving rise to the unitary similitude group $\mathbb{G}$ and simple Shimura varieties Sh_K as in [Kot92a], where $K \subset \mathbb{G}(\mathbb{A}_f)$ is sufficiently small compact open subgroup. Fix a prime number p . Let $\mathbb{E}$ be the reflex field and $\mathfrak{p}$ a prime of $\mathbb{E}$ above the rational prime p . We assume that all places of $\mathbb{F}_0$ above p are split in $\mathbb{F}$. In particular, $\mathbb{G}_{\mathbb{Q}_p}$ is a product of inner forms of Weil restrictions of general linear groups. Also we assume that $K = K^p K_p$ for compact open subgroups $K_p \subset \mathbb{G}(\mathbb{Q}_p)$ and $K^p \subset \mathbb{G}(\mathbb{A}_f^p)$.

Fix another prime number $\ell \neq p$ and let ξ be an algebraic representation of $\mathbb{G}$ giving rise to an ℓ -adic local system $\mathcal{F}_{\xi,K}$ on Sh_K . Consider the ℓ -adic étale cohomology groups

$$H_{\xi}^i := \varinjlim_K H_{\xi,t}^i(\mathrm{Sh}_K \otimes_{\mathbb{E}} \overline{\mathbb{Q}}, \mathcal{F}_{\xi,K})$$

and form their alternating sum $H_{\xi}^* := \sum_i (-1)^i H_{\xi}^i$ in the Grothendieck group of $\overline{\mathbb{Q}}_{\ell}$ -representations of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{E}) \times \mathbb{G}(\mathbb{A}_f)$. The main global theorem concerns the restriction of this representation to the local Weil group $W_{\mathbb{E}_p}$. We choose a smooth $\mathbb{Z}_p$ -model $\mathcal{G}$ of $\mathbb{G}_{\mathbb{Q}_p}$ such that $\mathcal{G}(\mathbb{Z}_p)$ is a parahoric subgroup of $\mathbb{G}(\mathbb{Q}_p)$, and we always assume $K_p \subseteq \mathcal{G}(\mathbb{Z}_p)$.

Theorem 1.1.1. *There is an identity in the Grothendieck group of $\mathbb{G}(\mathbb{A}_f^p) \times \mathcal{G}(\mathbb{Z}_p) \times W_{\mathbb{E}_p}$ representations*

$$H_{\xi}^* \cong \sum_{\pi_f} a(\pi_f) \pi_f \otimes (r_{-\mu} \circ \varphi_{\pi_p}|_{W_{\mathbb{E}_p}}) \cdot |^{-\dim \mathrm{Sh}/2}.$$

We refer to Theorem 4.3.1 for the more precise statement. Here we only mention that φ_{π_p} is the semisimple L -parameter of the irreducible smooth representation π_p of $\mathbb{G}(\mathbb{Q}_p)$. Note that by our assumption $\mathbb{G}_{\mathbb{Q}_p}$ is an inner form of a product of Weil restrictions of a general linear groups, hence the local Langlands correspondence is known for $\mathbb{G}_{\mathbb{Q}_p}$; see, e.g., [Coh18].

This result is proved by adapting Scholze's generalization of the Langlands-Kottwitz method to bad reduction cases as in [Sch13b]. Roughly speaking, using an $\mathcal{O}_{\mathbb{E}_p}$ -model of the Shimura variety with $\mathcal{G}(\mathbb{Z}_p)$ -level structure and the description of points in its special fiber, one gets the following expression from the Grothendieck-Lefschetz trace formula:

$$(1.1) \quad \mathrm{Tr}(\tau \times hf^p|H_{\xi}^*) = \sum_{(A,u,\lambda)} \mathrm{vol}(I(\mathbb{Q}) \backslash I(\mathbb{A}_f)) O_{\gamma}(f^p) TO_{\delta\sigma}(\phi_{\tau,h}) \mathrm{tr} \xi(\gamma_{\ell}).$$

Here $\tau \in \mathrm{Frob}^j I_{\mathbb{E}_p} \subset W_{\mathbb{E}_p}$ for some $j \geq 1$ (where $\mathrm{Frob} \in W_{\mathbb{E}_p}$ is a geometric Frobenius element and $I_{\mathbb{E}_p} \subset W_{\mathbb{E}_p}$ is the inertia subgroup), $h \in C_c^{\infty}(\mathcal{G}(\mathbb{Z}_p))$ is a cut-off function at p and $f^p \in C_c^{\infty}(\mathbb{G}(\mathbb{A}_f^p))$ is a function corresponding to a Hecke operator away from p . The sum runs over isomorphism classes of virtual polarized abelian varieties (A, u, λ) which, roughly speaking, parametrize isogeny classes in the set of κ_p -points of the special fiber of the integral model. For each isogeny class one has an associated σ -conjugacy class $\delta \in \mathbb{G}(\mathbb{Q}_{p^r})$ where $\mathbb{Q}_{p^r}$ is the degree $r := j[\kappa_p : \mathbb{F}_p]$ unramified extension of $\mathbb{Q}_p$ and σ is the Frobenius automorphism on $\mathbb{Q}_{p^r}$, and a conjugacy class $\gamma = (\gamma_{\ell})_{\ell \neq p} \in \mathbb{G}(\mathbb{A}_f^p)$. See §4.4.2 for details. The main ingredient in each summand is the twisted orbital integral of certain test function $\phi_{\tau,h} \in C_c^{\infty}(\mathbb{Q}_{p^r})$ that is defined in a purely local way using certain deformation spaces of p -divisible groups with extra structures.

Next one would like to relate the right hand side of (1.1) with the geometric side of the Arthur-Selberg trace formula for $\mathbb{G}$. The first step would be to associate a global (stable) conjugacy class $\gamma_0 \in \mathbb{G}(\mathbb{Q})$ to each summand so that $(\gamma_0; \gamma, \delta)$ forms a Kottwitz triple. However, when the local group $\mathbb{G}_{\mathbb{Q}_p}$ is not quasi-split, there is an obstruction to the existence of γ_0 . Namely, such a γ_0 exists if and only if the naive norm $N\delta := \delta\sigma(\delta) \cdots \sigma^{r-1}(\delta) \in \mathbb{G}(\mathbb{Q}_{p^r})$ is conjugate to an element in $\mathbb{G}(\mathbb{Q}_p)$; see Proposition 4.5.2. When $\mathbb{G}_{\mathbb{Q}_p}$ is quasi-split, as in [Kot92a], [Sch13b], and [SS13], this is always true and there is no obstruction to finding γ_0 . Now we can state our first local result, the aforementioned vanishing result.

Theorem 1.1.2. *With notations as above, if $N\delta \in \mathbb{G}(\mathbb{Q}_{p^r})$ is not $\mathbb{G}(\mathbb{Q}_{p^r})$ -conjugate to an element of $\mathbb{G}(\mathbb{Q}_p)$, then $TO_{\delta\sigma}(\phi_{\tau,h}) = 0$.*

The special case of this result where $\mathbb{G}(\mathbb{Q}_p)$ is the unit group of a division algebra and the relevant cocharacter is $(1, 0, \dots, 0)$ is proved in previous work of Shen ([She18, Theorem 5.4], [She19, Proposition 4.4]). Our approach to the general case is different from Shen's method. Vanishing statements like this were first conjectured by Rapoport, for test functions for certain Shimura varieties with parahoric level structure at p (see [Rap90, Conj. 5.7]). Special cases of his conjecture were proved by Rapoport in *loc. cit.*, and were later generalized by Waldspurger to some additional cases (private

communication), also by passing to the spectral side. Rapoport made a related vanishing conjecture in the framework of the Langlands-Rapoport Conjecture, see [Rap05, Conj. 10.2].

As an immediate consequence of Theorem 1.1.2, only those isogeny classes to which we can associate a Kottwitz triple contribute to the sum (1.1) and we get

$$\mathrm{Tr}(\tau \times hf^p | H_\xi^*) = \sum_{(\gamma_0; \gamma, \delta)} c(\gamma_0; \gamma, \delta) O_\gamma(f^p) TO_{\delta\sigma}(\phi_{\tau, h}) \mathrm{tr}\xi(\gamma_0).$$

To proceed further, we need to know the spectral information of the test function $\phi_{\tau, h}$. Let us introduce more notations. The Shimura data determines a conjugacy class of cocharacters μ of $\mathbb{G}(\mathbb{C})$ (whose field of definition is the reflex field $\mathbb{E}$) that we also view as characters of the maximal torus $\hat{\mathbb{T}}$ in the Langlands dual group $\hat{\mathbb{G}}$. Let $r_{-\mu}$ be the irreducible algebraic representation of $\hat{\mathbb{G}} \rtimes W_{\mathbb{E}_p}$ whose restriction to $\hat{\mathbb{G}}$ has extreme weight $-\mu$, such that $W_{\mathbb{E}_p}$ acts trivially on the extreme weight spaces. For each $\tau \in W_{\mathbb{E}_p}$, let $z_{\tau, -\mu}$ be the element of the (stable) Bernstein center of $\mathbb{G}_{\mathbb{Q}_p}$ that acts on any irreducible smooth representation π of $\mathbb{G}(\mathbb{Q}_p)$ by the scalar

$$\mathrm{Tr}(\tau | (r_{-\mu} \circ \varphi_\pi |_{W_{\mathbb{E}_p}}) | \cdot |_{\mathbb{E}_p}^{-\langle \rho, \mu \rangle})$$

where $\varphi_\pi : W_{\mathbb{Q}_p} \rightarrow \hat{\mathbb{G}} \rtimes \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ is the semisimple Langlands parameter of π and ρ is half the sum of positive roots of $\mathbb{G}$. Here the notion of positive root is determined by a choice of Borel subgroup $\mathbb{B}$ containing $\mathbb{T}$, and in forming $\langle \rho, \mu \rangle$ we also choose a $\mathbb{B}$ -dominant representative $\mu \in X_*(\mathbb{T})$ in its Weyl-group orbit. Now we can state our second local result.

Theorem 1.1.3. *The Scholze test function $\phi_{\tau, h} \in C_c^\infty(\mathbb{G}(\mathbb{Q}_{p^r}))$ has matching (twisted) orbital integrals with $z_{\tau, -\mu} * h \in C_c^\infty(\mathbb{G}(\mathbb{Q}_p))$.*

This establishes a key relation between the Scholze test functions and the stable Bernstein center. Similar relations of test functions for bad reduction with the stable Bernstein center were predicted by the second author and Kottwitz, and later by Scholze and Shin. We refer the reader to [Hai14] for some background on the test function conjecture and on the transfer conjectures for the stable Bernstein center; these ideas of the second author and Kottwitz were fleshed out during the period 1998-2011. Subsequently, Scholze and Shin formulated a more flexible version (in particular allowing for arbitrary cut-off functions h) in [SS13]. The version of Theorem 1.1.3 is adapted to the process of pseudostabilization which can be used in the setting of this article; the above references are in the framework of stabilization, which is more widely applicable. When $\mathbb{G}_{\mathbb{Q}_p}$ is quasi-split, Theorem 1.1.3 was proved in [SS13]. Our proof involves a global-local argument which has the same basis as the argument in [SS13], but which is more complicated both in terms of local harmonic analysis and the global geometric argument, when the local group is allowed to be non-quasi-split. In particular, for our argument it is necessary to introduce two companion global groups $\mathbb{G}_\beta$ and $\mathbb{G}'$ whose roles will be detailed later in this introduction. Roughly speaking, we will study the Shimura varieties attached to the group $\mathbb{G}_\beta$, and express the point-counting of their special fibers in terms of automorphic representations attached to the other group $\mathbb{G}'$ (instead of the group $\mathbb{G}_\beta$ itself).

In fact, for many cut-off functions h we can also find an element in the stable Bernstein center of $\mathbb{G}(\mathbb{Q}_{p^r})$ whose convolution with certain cut-off functions on $\mathbb{G}(\mathbb{Q}_{p^r})$ have matching twisted orbital integrals with $\phi_{\tau, h}$, see Corollary 3.4.5. As a consequence, we verify the special case of the test function conjecture of the second author and Kottwitz in our setting; in particular this expresses the semisimple trace of Frobenius in terms of certain functions in the Bernstein center:

$$\mathrm{Tr}^{\mathrm{ss}}(\mathrm{Frob}_p^j \times f^p | H^*(\mathrm{Sh}_{K_p K^p} \otimes_{\mathbb{E}} \bar{\mathbb{Q}}, \mathcal{F}_{\xi, K_p K^p})) = \sum_{(\gamma_0; \gamma, \delta)} c(\gamma_0; \gamma, \delta) O_\gamma(f^p) TO_{\delta\sigma}(z_{-\mu}^{(r)} * e_{K_{p^r}}) \mathrm{tr}\xi(\gamma_0).$$

Here the level $K_p \subset \mathcal{G}(\mathbb{Z}_p)$ at p can be arbitrarily small, $K_{p^r} \subset \mathbb{G}(\mathbb{Q}_{p^r})$ is a compact open subgroup corresponding to K_p in a suitable sense, and $e_{K_{p^r}}$ is the characteristic function of K_{p^r} divided by its volume. Further, $f^p \in C_c^\infty(K^p \backslash \mathbb{G}(\mathbb{A}_f^p)/K^p)$ is any function. Finally the main ingredient is the element $z_{-\mu}^{(r)}$ in the stable Bernstein center of $\mathbb{G}(\mathbb{Q}_{p^r})$, which is defined similarly to $z_{\tau, -\mu}$, using

the semisimple trace (which coincides with trace on inertia invariants in the current setting) instead of the usual trace. See Corollary 4.6.2 for a complete statement and section §4.6 for the details.

The proof of Corollary 3.4.5 involves more delicate harmonic analysis than what is needed for the proof of Theorem 1.1.3: it requires a result on the pseudostabilization base-change of the stable Bernstein center, namely Theorem 2.5.7, along with a key ingredient thereof, Proposition 2.5.5. We summarize the combination of these results as follows. Let $Z \in \mathfrak{Z}_{G(\mathbb{Q}_{p^r})}^{\text{st}}$ be any element in the stable Bernstein center, regarded as an element of the usual Bernstein center for $G(\mathbb{Q}_{p^r})$ (see §2.5). The element Z has a base-change $b(Z) \in \mathfrak{Z}_{G(\mathbb{Q}_p)}^{\text{st}}$, as constructed in [Hai14].

Theorem 1.1.4. *If $\phi \in C_c^\infty(G(\mathbb{Q}_{p^r}))$ satisfies the vanishing property, then so does the convolution $Z * \phi$. Letting $f \in C_c^\infty(G)$ denote the pseudostabilization base change transfer of ϕ (which exists by Lemma 2.5.4), the function $b(Z) * f$ is a pseudostabilization base change transfer of $Z * \phi$.*

The rather involved proof of this result relies on our twisted local Jacquet-Langlands correspondence (Theorem 2.4.1), which is also proved by a global method, and which to our knowledge is a novel result. We refer to §2 for the details. We expect that the obvious analogue of Theorem 1.1.4 for *stable orbital integrals* will be true in general, for all connected reductive groups over p -adic fields; this will hopefully be pursued in future extensions of our approach to all such groups.

Our debt to the ideas of Kottwitz, Rapoport, Harris-Taylor, Scholze, and Shin should be clear to the reader. Like the related articles [She18, Sch13b, SS13], our method builds on Rapoport's extension of the Langlands-Kottwitz method to bad reduction, which proposed the idea of counting points by weighting fixed points of Frobenius-Hecke correspondences by the (semisimple) trace of Frobenius on the stalk of the nearby cycles of appropriate local systems on the generic fiber (see [Rap05, §10]); but following Scholze, we avoid use of integral models except at the bottom of the tower, where we use an integral model for parahoric level structure. Further, like the earlier articles, we also rely heavily on the global-local arguments and Galois representation constructions of Harris-Taylor [HT01], in the sense that we generalize their arguments to work in our specific context.

1.2. Strategy of proof. Although the global result Theorem 1.1.1 will be an immediate consequence of the local results Theorem 1.1.2 and Theorem 1.1.3, our proof of the local results is by a global method. Starting with the original Shimura data $(\mathbb{G}, X)$, we construct a companion Shimura data $(\mathbb{G}_\beta, X_\beta)$ whose localization at p is the same as that for $(\mathbb{G}, X)$, and we show that in order to prove our local theorems, it is enough to work with $(\mathbb{G}_\beta, X_\beta)$ (see Proposition 3.3.3). More precisely, the group $\mathbb{G} = \text{GU}(\mathbb{D}, *)$ and the CM field $\mathbb{F}$ are replaced by another pair, which we denote by $\mathbb{G}_\beta$ and $\mathbb{F} = \mathbb{K}\mathbb{F}_0$. The latter $\mathbb{F}$ is a CM field containing a purely imaginary quadratic extension $\mathbb{K}$ in which p splits, and $\mathbb{G}_\beta = \text{GU}(\mathbb{D}_\beta, *_\beta)$ is a group like $\mathbb{G}$ whose signature at the archimedean places is determined by the original μ , as in our section 5.1.3 (especially Lemma 5.1.1). For this unitary group $\mathbb{G}_\beta$, we have the ambient group $\tilde{\mathbb{G}}_\beta$, the linear group of units of $\mathbb{D}_\beta$ over its center $\mathbb{F}$. We establish the required theory of base change for these unitary groups (Theorem 5.2.2) and use this to obtain the Galois representation of $\text{Gal}(\mathbb{F}/\mathbb{F})$ attached to a cuspidal automorphic representation π of $\mathbb{G}_\beta(\mathbb{A})$ (Theorem 5.3.1), which by construction satisfies global-local compatibility. For each relevant π we induce the corresponding Galois representation using Shapiro's Lemma to get something almost like a global Langlands parameter, namely an admissible homomorphism $\varphi_\ell(\pi) : \text{Gal}(\bar{\mathbb{Q}}/\mathbb{K}) \rightarrow {}^L\mathbb{G}_\beta(\bar{\mathbb{Q}}_\ell)$ (see the proof of Corollary 5.3.2). We restrict this further to $\text{Gal}(\bar{\mathbb{Q}}/\bar{\mathbb{E}})$ where $\bar{\mathbb{E}} = \mathbb{E}\mathbb{K}$ (composite of the reflex field with $\mathbb{K}$), in order to compare it with the Galois representation $\sigma_{\beta, \xi}(\pi_f)$ appearing tautologically in the cohomology of the Shimura variety attached to $\mathbb{G}_\beta$. Having done all that, we are in a position to use the results of Kottwitz at the good places, and the Chebotarev density theorem, in order to prove that Theorem 1.1.1 holds for $\text{Sh}(\mathbb{G}_\beta, X_\beta)$, even without using the Langlands-Kottwitz-Scholze method for this special Shimura variety. The details are given in the proof of Corollary 5.3.2.

For the next step, we run the Langlands-Kottwitz-Scholze method for $\text{Sh}(\mathbb{G}_\beta, X_\beta)$. This connects the purely local test function $\phi_{\tau, h}$ to global cohomology groups where the Galois action is now sufficiently pinned-down, and it would permit us to prove the desired properties of $\phi_{\tau, h}$, provided

that the Langlands-Kottwitz-Scholze method could be pushed to the point of stabilization. As mentioned above, there are obstructions to converting the resulting Lefschetz trace formula of form (1.1) into an Arthur-Selberg trace formula without knowing the validity of the vanishing property of the test function $\phi_{\tau,h}$ and without being able to construct, even when the vanishing property is known, the global elements γ_0 in the desired Kottwitz triples $(\gamma_0; \gamma, \delta)$. To surmount these obstructions, our main innovation is to find a replacement $(\gamma'_0; \gamma, \delta)$ of the usual Kottwitz triples using an auxiliary group $\mathbb{G}'$, also constructed from a division algebra with involution of the second kind, which is quasi-split at p and isomorphic to $\mathbb{G}_\beta$ away from p and ∞ . With this modification we can relate the right hand side of (1.1) to a simple trace formula for $\mathbb{G}'$. Finally, by comparing simple trace formulas for $\mathbb{G}_\beta$ and $\mathbb{G}'$, we get the local results Theorems 1.1.2 and 1.1.3 and hence we deduce the global result Theorem 1.1.1 in full generality.

1.3. Organization of the article. In §2, we establish several results in p -adic harmonic analysis. After reviewing various notions of transfer of local test functions, we prove a twisted version of the local Jacquet-Langlands correspondence (Theorem 2.4.1) using a global method involving twisted trace formulae. As a consequence we deduce the technical results mentioned above concerning base change of the stable Bernstein center and the permanence of the vanishing property.

In §3, we study the deformation spaces of p -divisible groups with EL structures and use it to define the local test functions $\phi_{\tau,h}$, generalizing the constructions in [Sch13b]. Then we state our local results concerning its vanishing property (Theorem 3.4.3) and its relation with certain elements in the stable Bernstein center of the local group (Theorem 3.4.4).

In §4, we first review the construction of Kottwitz type simple Shimura varieties and their integral models with parahoric level structures. Then we state the main global result Theorem 4.3.1 giving the description of the cohomology, review the Langlands-Kottwitz-Scholze method, and explain the extra ingredient needed to adapt it to our situation, that is, the vanishing property Theorem 3.4.3 of the local test function $\phi_{\tau,h}$. After that we deduce Theorem 4.3.1 from the main local result Theorem 3.4.4 concerning the spectral information of the test function $\phi_{\tau,h}$.

In §5, we prove the local results by embedding the local situation into a global simple Shimura variety, for which we can use several results on automorphic representations to establish the description of its cohomology directly, without using the Langlands-Kottwitz-Scholze method. From there we deduce the local results concerning the test function $\phi_{\tau,h}$ by modifying the Langlands-Kottwitz-Scholze method using an auxiliary group $\mathbb{G}'$.

1.4. Notations. Since we work in both local and global setting and will pass back and forth between them, to minimize confusion we distinguish different settings by different fonts. We use Roman letters F, D, B, G etc. to denote objects over the local field $\mathbb{Q}_p$. But note that in §2 we use the boldface letter $\mathbf{G}$ to denote an algebraic group over $\mathbb{Q}_p$, whereas we use the Roman letter $G = \mathbf{G}(\mathbb{Q}_p)$ to denote its set of $\mathbb{Q}_p$ -points. We use the calligraphic font $\mathcal{G}$ to denote a group scheme over $\mathbb{Z}_p$, while blackboard bold letters $\mathbb{F}, \mathbb{D}, \mathbb{B}, \mathbb{G}$ etc. are used to denote objects defined over the global field $\mathbb{Q}$.

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2. PREPARATIONS IN LOCAL HARMONIC ANALYSIS

In this section we let $\mathbf{G}$ be a reductive algebraic group over $\mathbb{Q}_p$ whose quasi-split inner form $\mathbf{G}^*$ is isomorphic to a product of Weil restrictions of general linear groups. Let $r \geq 1$ be an integer. Let $\mathbf{G}_r := \text{Res}_{\mathbb{Q}_{p^r}/\mathbb{Q}_p} \mathbf{G}_{\mathbb{Q}_{p^r}}$, and let $\mathbf{G}_r^* := \text{Res}_{\mathbb{Q}_{p^r}/\mathbb{Q}_p} \mathbf{G}_{\mathbb{Q}_{p^r}}^*$ be the quasi-split inner form

of $\mathbf{G}_r$. Fix a generator σ of the cyclic group $\text{Gal}(\mathbb{Q}_{p^r}/\mathbb{Q}_p)$ which induces automorphisms of the $\mathbb{Q}_p$ -algebraic groups $\mathbf{G}_r$ and $\mathbf{G}_r^*$ that we also denote by σ , respectively σ^* . We denote $G = \mathbf{G}(\mathbb{Q}_p)$, $G_r = \mathbf{G}_r(\mathbb{Q}_p) = \mathbf{G}(\mathbb{Q}_{p^r})$, and similarly $G^* = \mathbf{G}^*(\mathbb{Q}_p)$, $G_r^* = \mathbf{G}_r^*(\mathbb{Q}_p) = \mathbf{G}^*(\mathbb{Q}_{p^r})$.

2.1. Transfer of twisted orbital integrals. For each element $\delta \in G_r$, define its *naive norm* to be the element

$$N_r \delta := \delta \sigma(\delta) \cdots \sigma^{r-1}(\delta) \in G_r.$$

Then there exists an element $\mathcal{N}_r \delta \in G^*$ that is stably conjugate to the naive norm $N_r \delta$. The stable conjugacy class of $\mathcal{N}_r \delta$ only depends on the stable σ -conjugacy class of δ and we call it *the norm of δ in G^** ; see [Kot82, §5]. Similarly, for $\delta \in G_r^*$ we have the naive norm $N_r^* \delta \in G_r^*$ and the norm $\mathcal{N}_r^* \delta \in G^*$, the latter of which is, a priori, a stable conjugacy class in G^* . We say $\delta \in G_r$ (resp. $\delta^* \in G_r^*$) is σ -semisimple if $N_r \delta$ (resp. $N_r^* \delta^*$) is semisimple.

For the groups and elements considered here, stable (σ -) conjugacy just coincides with ordinary (σ -) conjugacy.

Lemma 2.1.1. (1) *The map $\delta \mapsto \mathcal{N}_r \delta$ defines an injection from the set of σ -conjugacy classes of σ -semisimple elements of G_r to the set of conjugacy classes of semisimple elements of G^* . A similar statement holds for G_r^* .*
 (2) *For each $\delta \in G_r$ such that $N_r \delta$ is G_r -conjugate to a regular semi-simple element in G , there exists $\delta^* \in G_r^*$ such that $N_r \delta = \mathcal{N}_r^* \delta^*$.*

Proof. The first part follows from [AC89, Lemma 1.1] (which is stated for G_r^* but the proof also works for G_r). In more detail, by [Kot82, Cor. 5.3], the map $\delta \mapsto N_r \delta$ induces an injection from σ - $\mathbf{G}_r(\overline{\mathbb{Q}_p})$ conjugacy classes in $\mathbf{G}_r(\overline{\mathbb{Q}_p})$ represented by $\mathbf{G}_r(\mathbb{Q}_p) = \mathbf{G}(\mathbb{Q}_{p^r})$ to conjugacy classes in $\mathbf{G}^*(\overline{\mathbb{Q}_p})$ represented by $\mathbf{G}^*(\mathbb{Q}_p)$. It is enough to prove that if $x, y \in \mathbf{G}(\mathbb{Q}_{p^r})$ are σ - $\mathbf{G}_r(\overline{\mathbb{Q}_p})$ -conjugate and are σ -semisimple, then they are σ - $\mathbf{G}(\mathbb{Q}_{p^r})$ -conjugate (see the end of [Kot82, §5]). Viewing the σ -centralizer $\mathbf{G}_{x\sigma}$ as a $\mathbb{Q}_p$ -subgroup of $\mathbf{G}_r$, it is enough to show that

$$\ker(H^1(\mathbb{Q}_p, \mathbf{G}_{x\sigma}) \rightarrow H^1(\mathbb{Q}_p, \mathbf{G}_r))$$

is trivial. In fact we will show that $H^1(\mathbb{Q}_p, \mathbf{G}_{x\sigma})$ is trivial. Recall that $\mathbf{H} \mapsto H^1(\mathbb{Q}_p, \mathbf{H})$ is invariant under inner twists of a connected reductive $\mathbb{Q}_p$ -group $\mathbf{H}$; see [Kot86b, Thm. 1.2]. By [Kot82, Lem. 5.8], $\mathbf{G}_{x\sigma}$ is a $\mathbb{Q}_p$ -inner form of the centralizer $\mathbf{G}_{\mathcal{N}_r x}^*$, and the latter is connected and reductive (using that $\mathcal{N}_r x$ is semisimple and $\mathbf{G}_{\text{der}}^* = \mathbf{G}_{\text{sc}}^*$). Hence it is enough to prove that $H^1(\mathbb{Q}_p, \mathbf{G}_{\mathcal{N}_r x}^*)$ is trivial. But as $\mathcal{N}_r x$ is semisimple, the group $\mathbf{G}_{\mathcal{N}_r x}^*$ is the group of units of a semisimple $\mathbb{Q}_p$ -algebra, and so by a version of Hilbert's Theorem 90, $H^1(\mathbb{Q}_p, \mathbf{G}_{\mathcal{N}_r x}^*)$ is trivial. This proves the assertion for G_r , and the proof for G_r^* is similar.

For the second part, we argue as in [AC89, Lemma 1.3]. By construction G (resp. G^*) is the unit group of a semisimple $\mathbb{Q}_p$ -algebra B (resp. B^*). Note that B^* is a product of matrix algebras. Let $B_r := B \otimes_{\mathbb{Q}_p} \mathbb{Q}_{p^r}$ and $B_r^* := B^* \otimes_{\mathbb{Q}_p} \mathbb{Q}_{p^r}$. Let $u = N_r \delta = \delta \sigma(\delta) \cdots \sigma^{r-1}(\delta)$. By assumption, there exists $h \in G_r$ such that $huh^{-1} \in G$. Note that $huh^{-1} = N_r(h\delta\sigma(h)^{-1})$ so replacing δ by $h\delta\sigma(h)^{-1}$ we may assume that $u \in G$. Let L be the centralizer of u in B . Then $L_r := L \otimes_{\mathbb{Q}_p} \mathbb{Q}_{p^r}$ is the centralizer of u in B_r . Since $\delta^{-1}u\delta = \sigma(u) = u$, we have $\delta \in L_r$. Choose an embedding of $\mathbb{Q}_p$ -algebras $L \hookrightarrow B^*$ and let u^* be the image of u . Let δ^* be the image of δ under the resulting $\mathbb{Q}_{p^r}$ -algebra embedding $L_r \hookrightarrow B_r^*$. Then $u^* \in G^*$ is stably conjugate to u and since $N_{L_r/L}(\delta) = N_r(\delta) = u$ we get $N_r^*(\delta^*) = N_{L_r/L}(\delta^*) = u^*$. $\square$

Remark 2.1.2. For a general $\delta \in G_r$, there may not exist $\delta^* \in G_r^*$ such that $\delta \leftrightarrow \delta^*$. We give an example when $r = 2$. Let $F/\mathbb{Q}_p$ be a finite extension and $\varpi \in F$ be a uniformizer. Let F_2 be the degree 2 unramified extension of F . Assume that $F_2 = F \otimes_{\mathbb{Q}_p} \mathbb{Q}_{p^2}$. Let $\Pi = \begin{bmatrix} 0 & 1 \\ \varpi & 0 \end{bmatrix}$ and consider the quaternion division algebra

$$B = \{x \in M_2(F_2), x = \Pi^{-1}\sigma^*(x)\Pi\}$$

Then $B_2 = B \otimes_{\mathbb{Q}_p} \mathbb{Q}_{p^2} \cong M_2(F_2) = B_2^*$. For an element $u = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \in B^* = M_2(F)$ with $ab \neq 0$ and $a \neq b$, u is a norm from $B_2^* = M_2(F_2)$ if and only if $a, b \in N_{F_2/F} F_2^\times$.

Take $\delta = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} \Pi \in B_2$ with $\alpha\beta \neq 0$ and $\alpha \neq \beta$. Then we get

$$N_2\delta = \delta\sigma(\delta) = \delta\Pi^{-1}\sigma^*(\delta)\Pi = \begin{bmatrix} \varpi\alpha\sigma(\alpha) & 0 \\ 0 & \varpi\beta\sigma(\beta) \end{bmatrix} \in M_2(F).$$

Then $N_2(\delta)$ is not a norm from B_2^* (and not conjugate to any element in B).

Definition 2.1.3. For any $\phi \in C_c^\infty(G_r)$, we say that a function $\phi^* \in C_c^\infty(G^*)$ is a *stable base change transfer* of $\phi \in C_c^\infty(G_r)$ if for any $\delta \in G_r$ with semisimple norm $\mathcal{N}_r\delta =: \gamma^* \in G^*$ we have $e(G_{\delta\sigma})TO_{\delta\sigma}(\phi) = e(G_{\gamma^*})O_{\gamma^*}(\phi^*)$, whereas if a semisimple element $\gamma^* \in G^*$ is not conjugate to a norm from G_r , then $O_{\gamma^*}(\phi^*) = 0$. Here $e(G_{\delta\sigma})$ is the Kottwitz sign of the σ -centralizer of δ , viewed as a group over $\mathbb{Q}_p$. (See §4.5.3 for some reminders about the Kottwitz sign.)

Remark 2.1.4. In making this definition, we choose Haar measures on G_r , $G_{\delta\sigma}$, G^* , $G_{\gamma^*}^*$, such that the measures on $G_{\delta\sigma}$ and its inner form $G_{\gamma^*}^*$ are compatible. The notion of matching depends on these choices.

Remark 2.1.5. Since our group G is an inner form of a product of Weil restrictions of GL_n , stable conjugacy coincides with conjugacy due to vanishing of Galois cohomology as in the proof of Lemma 2.1.1. This justifies that our notion of transfer is compatible with the general definition as in [Lab99, §3.2], for example.

The existence of stable base change in general is established in [Lab99, §3.3.1]. It is a special instance of twisted endoscopic transfer.

The following definition plays a key role in this article.

Definition 2.1.6. We say that a function $\phi \in C_c^\infty(G_r)$ has the *vanishing property* if $TO_{\delta\sigma}(\phi) = 0$ for any σ -semisimple element $\delta \in G_r$ such that $N_r\delta$ is not G_r -conjugate to an element in G .

2.2. Reduction to matching at regular elements. It is well-known principle that in order to check the matching of (twisted) orbital integrals on all semisimple elements, it should suffice to check the matching on *regular* semisimple elements. We include a proof of this fact in our setting, following the arguments of [Clo90, Prop. 7.2].

Proposition 2.2.1. *Let $\phi \in C_c^\infty(G_r)$ and $\phi^* \in C_c^\infty(G^*)$. Suppose that for any regular semisimple element $\gamma^* \in G^*$ we have $O_{\gamma^*}(\phi^*) = 0$ if γ^* is not a norm from G_r , while if $\gamma^* = \mathcal{N}_r\delta$ we have*

$$O_{\gamma^*}(\phi^*) = TO_{\delta\sigma}(\phi).$$

Then ϕ^ is a stable base change transfer of ϕ at all semisimple elements γ^* . (Note that the usual sign $e(\mathbf{G}_{\delta\sigma})$ which would appear is trivial under our assumption that γ^* is regular semisimple, since $\mathbf{G}_{\delta\sigma}$ is a torus.)*

Proof. We start by recalling some facts about Shalika germs. Suppose $\gamma^* = \mathcal{N}_r\delta$ is a semisimple element in G^* , not necessarily regular. By [Kot82, Lem. 5.8], once we fix a $\mathbb{Q}_p$ -inner twisting $\psi : \mathbf{G} \rightarrow \mathbf{G}^*$ there is an $\mathbb{Q}_p$ -inner twisting $\mathbf{G}_{\delta\sigma} \rightarrow \mathbf{G}_{\gamma^*}^*$, canonical up to inner automorphisms by $\mathbf{G}_{\gamma^*}^*(\bar{\mathbb{Q}}_p)$. For any maximal $\mathbb{Q}_p$ -torus $\mathbf{T} \subset \mathbf{G}_{\delta\sigma}$ ($\mathbf{T}$ need not be $\mathbb{Q}_p$ -elliptic), we may assume our chosen inner twisting ψ gives a $\mathbb{Q}_p$ -embedding $\psi : \mathbf{T} \xrightarrow{\sim} \mathbf{T}^* \hookrightarrow \mathbf{G}_{\gamma^*}^* \subset \mathbf{G}^*$, by [Kot82, Cor. 2.2]. We can arrange (by conjugating ψ by $\mathbf{G}^*(\bar{\mathbb{Q}}_p)$) that $\psi(N_r\delta) = \gamma^*$. Then we can further assume (conjugating ψ if necessary by an element in $\mathbf{G}_{\gamma^*}^*(\bar{\mathbb{Q}}_p)$ and using [Kot82, Cor. 2.2]) that ψ gives a $\mathbb{Q}_p$ -isomorphism $\psi : \mathbf{T} \xrightarrow{\sim} \mathbf{T}^* \subset \mathbf{G}_{\gamma^*}^*$. Then we have $N_r\delta \in T$ and $\psi(N_r\delta) = \gamma^*$.

Fix a Haar measure dt on $T := \mathbf{T}(\mathbb{Q}_p)$. Let $\{u_j\}$ be a set of representatives of unipotent conjugacy classes in $G_{\delta\sigma}$. It follows from [Rog90, Proposition 8.1.1] that there is a sufficiently small neighborhood $0 \in \Omega \subset \mathrm{Lie}(T)$ such that

- the exponential map defines an isomorphism from Ω to an open neighborhood of 1 in T ;
- Let $\Omega' \subset \Omega$ be the open subset consisting of elements $X \in \Omega$ such that $\exp(X) \in T$ is regular. Then $a\Omega' \subset \Omega'$ for any $0 \neq a \in \mathbb{Z}_p$;
- The germ expansion holds on Ω' , i.e. for all $0 \neq t \in \mathbb{Z}_p$ and any $X \in \Omega'$, denote $x_t := \exp(tX)$, then we have

$$TO_{x_t\delta\sigma}(\phi) = \sum_j \Gamma_{u_j}^G(x_t\delta) TO_{u_j\delta\sigma}(\phi), \quad \forall \phi \in C_c^\infty(G_r)$$

where the Shalika germ Γ_{u_j} is a function on $\exp(\Omega')\delta$ that depends on the Haar measure on $T = G_{x_t\delta\sigma} = (G_{\delta\sigma})_{x_t}$ and the Haar measure on $G_{u_j\delta\sigma}$ used to define the orbital integrals.

By [Rog90, Proposition 8.1.2(b)], the function $\Gamma_{u_j}^G$ is homogeneous of degree $\frac{1}{2}d_{u_j} := \frac{1}{2} \dim(G_{\delta\sigma}/G_{u_j\delta\sigma})$ in the sense that for all $X \in \Omega'$ and $0 \neq t \in \mathbb{Z}_p$ we have

$$\Gamma_{u_j}^G(\exp(tX)\delta) = |t|^{-\frac{1}{2}d_{u_j}} \Gamma_{u_j}^G((\exp X)\delta).$$

There is a similar Shalika germ expansion

$$O_{\psi(\exp(rtX))\gamma^*}(\phi^*) = \sum_{j'} \Gamma_{u_{j'}}^{G^*}(\psi(\exp(rtX))\gamma^*) O_{u_{j'}^*, \gamma^*}(\phi^*), \quad \forall 0 \neq t \in \mathbb{Z}_p.$$

Here $\{u_{j'}\}_{j'}$ ranges over the set of unipotent conjugacy classes in $G_{\gamma^*}^*$, and the other notation is as before. Like the above, the Shalika germs here enjoy a similar homogeneity property.

Now we prove the vanishing statement. From the definition we verify easily that for any $X \in \Omega'$ and $0 \neq t \in \mathbb{Z}_p$ we have $N_r(\exp(tX)\delta) = \exp(rtX)N_r\delta$ and hence under our embedding $T \hookrightarrow G_{\gamma^*}^*$ we have $\mathcal{N}_r(\exp(tX)\delta) = \psi(\exp(rtX))\gamma^*$. Therefore if the semisimple element $\gamma^* \in G^*$ is not a norm from G_r , then $\psi(\exp(rtX))\gamma^*$ is also not a norm from G_r . Moreover for fixed γ^* , we may shrink the open set Ω' to ensure that $\psi(\exp(rtX))\gamma^*$ is regular for all $X \in \Omega'$. Then by the definition of stable base change transfer we have $O_{\psi(\exp(rtX))\gamma^*}(\phi^*) = 0$. Looking at the degree 0 term in the germ expansion we get that $O_{\gamma^*}(\phi^*) = 0$.

Next we again assume that $\gamma^* = \mathcal{N}_r\delta$ is a norm. We consider ψ , $\mathbf{T}$, and $\mathbf{T}^*$ as above. We assume moreover that $\mathbf{T}$ is a $\mathbb{Q}_p$ -elliptic torus in $\mathbf{G}_{\delta\sigma}$. Then $\mathbf{T}^*$ is also a $\mathbb{Q}_p$ -elliptic torus in the inner form $\mathbf{G}_{\gamma^*}^*$ of $\mathbf{G}_{\delta\sigma}$.

By Harish-Chandra descent of (twisted) orbital integrals, there exists $\phi_1 \in C_c^\infty(G_{\delta\sigma})$ and $\phi_1^* \in C_c^\infty(G_{\gamma^*}^*)$ such that for all $X \in \Omega'$ and $0 \neq t \in \mathbb{Z}_p$ sufficiently close to 0 we have

$$(2.1) \quad TO_{\exp(tX)\delta\sigma}^G(\phi) = O_{\exp(tX)\delta\sigma}^{G_{\delta\sigma}}(\phi_1), \quad O_{\psi(\exp(rtX))\gamma^*}(\phi^*) = O_{\psi(\exp(rtX))\gamma^*}^{G_{\gamma^*}^*}(\phi_1^*).$$

See, for example, the proof of [Rog90, p.114, proof of Proposition 8.1.2], or [AC89, p.23-24].

By the descent property of Shalika germs (see [Rog90, Proposition 8.1.2(a)]) we have

$$(2.2) \quad \Gamma_1^G((\exp(tX))\delta) = \Gamma_1^{G_{\delta\sigma}}(\exp(tX)), \quad \Gamma_1^{G^*}(\psi(\exp(rtX))\gamma^*) = \Gamma_1^{G_{\gamma^*}^*}(\psi(\exp(rtX))).$$

Comparing the degree 0 terms of the germ expansions in (2.1) we get

$$(2.3) \quad TO_{\delta\sigma}(\phi) = \phi_1(1), \quad O_{\gamma^*}(\phi^*) = \phi_1^*(1).$$

On the other hand, by assumption we have $TO_{\exp(tX)\delta\sigma}^{G_r}(\phi) = O_{\psi(\exp(rtX))\gamma^*}(\phi^*)$ and hence by (2.1) we get

$$O_{\exp(tX)\delta\sigma}^{G_{\delta\sigma}}(\phi_1) = O_{\psi(\exp(rtX))\gamma^*}^{G_{\gamma^*}^*}(\phi_1^*).$$

Then looking at the degree 0 terms of the germ expansion and using (2.2), (2.3) we get

$$\Gamma_1^{G_{\delta\sigma}}(\exp X) TO_{\delta\sigma}(\phi) = \Gamma_1^{G_{\gamma^*}^*}(\psi(\exp(rtX))) O_{\gamma^*}(\phi^*)$$

We note that $\mathbf{G}_{\gamma^*}^*$ is the quasi-split inner form of $\mathbf{G}_{\delta\sigma}$. Now we claim: since $\mathbf{T}$ is elliptic in $\mathbf{G}_{\delta\sigma}$, $\Gamma_1^{G_{\delta\sigma}}$ and $\Gamma_1^{G_{\gamma^*}^*}$ are constant functions of X and they differ by the factor $e(\mathbf{G}_{\delta\sigma})$. Then we would get that $O_{\gamma^*}(\phi^*) = e(\mathbf{G}_{\delta\sigma}) TO_{\delta\sigma}(\phi)$.

It remains to prove the claim above. Since $\mathbf{T}$ is $\mathbb{Q}_p$ -elliptic in $\mathbf{G}_{\delta\sigma}$, the main theorem of [Rog81] shows that

$$\Gamma_1^{G_{\delta\sigma}}(\exp X) = (-1)^{q(G_{\delta\sigma})} d(G_{\delta\sigma})^{-1}, \quad \forall X \in \Omega'$$

where $q(G_{\delta\sigma})$ is the $\mathbb{Q}_p$ -rank of the derived group of $\mathbf{G}_{\delta\sigma}$ and $d(G_{\delta\sigma})$ is the formal degree of the Steinberg representation of $G_{\delta\sigma}$ (with respect to a chosen Haar measure on $G_{\delta\sigma}$). There is a similar result for $\Gamma_1^{G_{\gamma^*}}$. In [Kot88], the measures on the groups are determined from Euler-Poincaré measures on the adjoint groups, in which case $d(G_{\gamma^*}) = (-1)^{q(G_{\gamma^*})}$ by [Bor76] (see also [GR10, §7.1]). Then by [Kot88, Theorem 1], if the measures on $G_{\delta\sigma}$ and G_{γ^*} are chosen as in [Kot88], we have $d(G_{\delta\sigma}) = d(G_{\gamma^*})$. Finally we note that since $\mathbf{G}_{\gamma^*}^*$ is the quasi-split inner form of $\mathbf{G}_{\delta\sigma}$, we have $e(\mathbf{G}_{\gamma^*}^*) = 1$ and $e(\mathbf{G}_{\delta\sigma}) = (-1)^{q(G_{\delta\sigma}) - q(G_{\gamma^*})}$. This completes the proof of the claim, and hence of the proposition. $\square$

2.3. A reinterpretation of stable base change transfer. When $\phi^* \in C_c^\infty(G^*)$ is the stable base change transfer of a function $\phi_r \in C_c^\infty(G_r)$, we will write this as $\phi_r \longleftrightarrow \phi^*$. When ϕ^* is the usual base-change of a function $\phi_r^* \in C_c^\infty(G_r^*)$, then we write $\text{BC}(\phi_r^*) = \phi^*$. (By “usual base-change”, we just mean what is described in Definition 2.5.3 below, when all groups in sight are quasi-split.) We now define a variant of the first relation.

Definition 2.3.1. For $\phi_r \in C_c^\infty(G_r)$ and $\phi_r^* \in C_c^\infty(G_r^*)$, we write $\phi_r \xrightarrow{\sigma\text{-reg}} \phi_r^*$ when the following property holds for every σ -regular semisimple element $\delta^* \in G_r^*$:

- If there is a $\delta \in G_r$ such that $N_r\delta = \mathcal{N}_r\delta^*$, then $TO_{\delta^*\sigma}^{G_r^*}(\phi_r^*) = TO_{\delta\sigma}^{G_r}(\phi_r)$.
- If there is no such δ , then $TO_{\delta^*\sigma}^{G_r^*}(\phi_r^*) = 0$.

Lemma 2.3.2. Fix $\phi_r \in C_c^\infty(G_r)$ and $\phi^* \in C_c^\infty(G^*)$.

- (1) If $\phi_r \longleftrightarrow \phi^*$ and ϕ_r has the vanishing property, then there exists a function $\phi_r^* \in C_c^\infty(G_r^*)$ such that $\phi_r \xrightarrow{\sigma\text{-reg}} \phi_r^*$ and $\text{BC}(\phi_r^*) = \phi^*$.
- (2) Conversely, suppose that ϕ_r has the vanishing property. If there exists a ϕ_r^* with $\phi_r \xrightarrow{\sigma\text{-reg}} \phi_r^*$ and $\text{BC}(\phi_r^*) = \phi^*$, then $\phi_r \longleftrightarrow \phi^*$.
- (3) If $G_r^* = (G^*)^r$, and if $\phi^* \in C_c^\infty(G^*)$ is a Jacquet-Langlands transfer of $\phi \in C_c^\infty(G)$, then ϕ^* is a stable base change transfer of $\phi_r := \phi \otimes e \otimes \cdots \otimes e \in C_c^\infty(G^r)$, where $e \in C_c^\infty(G)$ is an idempotent such that $\phi * e = \phi$.

Proof. (1). Suppose $\phi_r \longleftrightarrow \phi^*$ and ϕ_r has the vanishing property. We first claim that $\phi^* = \text{BC}(\phi_r^*)$ for some $\phi_r^* \in C_c^\infty(G_r^*)$. Suppose $\gamma^* \in G^*$ is regular semisimple. Suppose γ^* is not a norm from G_r^* . If $N_r\delta$ were G_r -conjugate to an element in $G(F)$, then Lemma 2.1.1(2) would imply that γ^* is a norm from G_r^* , a contradiction. Hence $N_r\delta$ is not G_r -conjugate to an element in $G(F)$, so that $TO_{\delta\sigma}(\phi_r) = 0$, by the vanishing property. Hence $O_{\gamma^*}^{G^*}(\phi^*) = 0$. This proves the claim thanks to [AC89, Prop. I.3.1]. Next we claim that $\phi_r \xrightarrow{\sigma\text{-reg}} \phi_r^*$. Suppose that $\delta^* \in G_r^*$ is σ -regular semisimple, and write $\mathcal{N}_r\delta^* = \gamma^* \in G^*$. If there is a $\delta \in G_r$ with $\mathcal{N}_r\delta = \mathcal{N}_r\delta^*$, then we have

$$TO_{\delta\sigma}^{G_r}(\phi_r) = O_{\gamma^*}^{G^*}(\phi^*) = TO_{\delta^*\sigma}^{G_r^*}(\phi_r^*),$$

as desired. Now suppose that there is no such δ . We have $TO_{\delta^*\sigma}(\pi_r^*) = O_{\gamma^*}(\phi^*)$. If γ^* is not a norm from G_r , then the latter vanishes, as desired. But γ^* cannot be a norm from G_r , since otherwise we would have $\mathcal{N}_r\delta^* = \gamma^* = \mathcal{N}_r\delta$ for some $\delta \in G_r$, contrary to assumption. This completes the proof of (1).

(2). Suppose that ϕ^*, ϕ_r^*, ϕ_r satisfy the given hypotheses. We need to verify the condition defining $\phi_r \longleftrightarrow \phi^*$ at any semisimple element $\gamma^* \in G^*$. By Proposition 2.2.1, we may assume γ^* is regular. Suppose $\gamma^* = \mathcal{N}_r\delta$ for $\delta \in G_r$. Suppose first that $\gamma^* = \mathcal{N}_r\delta^*$ for some $\delta^* \in G_r^*$. Then our hypotheses give

$$O_{\gamma^*}^{G^*}(\phi^*) = TO_{\delta^*\sigma}^{G_r^*}(\phi_r^*) = TO_{\delta\sigma}^{G_r}(\phi_r).$$

Next suppose that γ^* is not a norm from G_r^* . Then we have $O_{\gamma^*}(\phi^*) = 0$, using $\text{BC}(\phi_r^*) = \phi^*$. By Lemma 2.1.1(2), $N_r\delta$ cannot be G_r -conjugate to an element in G , and hence $TO_{\delta\sigma}(\phi_r) = 0$ by the vanishing property. Thus $O_{\gamma^*}(\phi^*) = TO_{\delta\sigma}(\phi_r)$ in all cases, when $\mathcal{N}_r\delta = \gamma^*$.

Now suppose that γ^* is not a norm from G_r . We claim that $O_{\gamma^*}(\phi^*) = 0$. If γ^* is not a norm from G_r^* , we are done using $\phi^* = \text{BC}(\phi_r^*)$. If $\gamma^* = \mathcal{N}_r\delta^*$, then this δ^* does not come from any $\delta \in G_r$, and hence $O_{\gamma^*}(\phi^*) = TO_{\delta^*\sigma}(\phi_r^*) = 0$, by the assumption $\phi_r \xrightarrow{\sigma\text{-reg}} \phi_r^*$, as desired.

(3). Define $\phi_r^* = \phi^* \otimes e^* \otimes \cdots \otimes e^*$, where $e^* \in C_c^\infty(G^*)$ is an idempotent with $\phi^* * e^* = \phi^*$. By the proof of [AC89, §I.5], we see that $\phi_r^* \xrightarrow{\sigma\text{-reg}} \phi_r$ in the obvious sense adapted to functions in $C_c^\infty(G^{*r})$ resp., $C_c^\infty(G^r)$. By [AC89, §I.5] again, we also have $\text{BC}(\phi_r^*) = \phi^*$, again in the obvious sense for the groups in question. In this situation, the naive norm map sends $(x_1, x_2, \dots, x_r)$ to $(x_1x_2 \cdots x_r, x_2x_3 \cdots x_rx_1, \dots)$ and it is easy to see that this is G_r -conjugate to the element $x_1x_2 \cdots x_r$ of G , embedded diagonally in G_r . Thus, the hypotheses of Lemma 2.1.1(2) are satisfied, meaning that for any regular semisimple $\gamma^* \in G^*$ which is a norm from G_r , it is also a norm from G_r^* . Then, by the same formal argument as in (2), the assertion follows. $\square$

Corollary 2.3.3. *If a function $\phi^* \in C_c^\infty(G^*)$ is a stable base change transfer of a function on G_r that has the vanishing property, it is also a stable base change transfer of a function on G_r^* .*

2.4. Local stable base change.

Theorem 2.4.1. *Let π^* be an irreducible tempered representation of G^* and let Π^* be the base change lift of π^* , which is by definition a σ -stable representation of G_r^* with canonical intertwining operator I_σ^* , see [AC89, §I.2].*

- (1) *If Π^* has a Langlands-Jacquet transfer to an irreducible tempered representation Π of G_r , then Π is σ -stable and we can choose an intertwining operator I_σ on Π such that for any $\phi \in C_c^\infty(G_r)$ with stable base change transfer $\phi^* \in C_c^\infty(G^*)$ we have*

$$\text{Tr}(\phi I_\sigma | \Pi) = e(\mathbf{G}_r) \text{Tr}(\phi^* | \pi^*).$$

- (2) *If Π^* does not have a Langlands-Jacquet transfer to G_r , then for any $\phi \in C_c^\infty(G_r)$ with stable base change transfer $\phi^* \in C_c^\infty(G^*)$ we have*

$$\text{Tr}(\phi^* | \pi^*) = 0.$$

2.4.1. Remark on σ -stability. Suppose Π is any irreducible tempered representation of G_r with Jacquet-Langlands transfer the irreducible tempered representation Π^* of G_r^* . Then Π is σ -stable if Π^* is σ^* -stable (the latter being automatic if Π^* is the base-change of an irreducible tempered representation π^* of G^*). To see this, we interpret σ -stability in terms of Harish-Chandra characters: Π is σ -stable if and only if $\Theta_\Pi(\sigma(\gamma_r)) = \Theta_\Pi(\gamma_r)$, for every regular semisimple $\gamma_r \in G_r$; similar statements hold for Π^* and Θ_{Π^*} and regular semisimple elements $\gamma_r^* \in G_r^*$. Moreover, the transfer relation $\gamma_r \mapsto \gamma_r^*$ is equivalent to $\sigma(\gamma_r) \mapsto \sigma^*(\gamma_r^*)$. So $\Pi^* \cong \Pi^* \circ \sigma^*$ implies the following for all $\gamma \mapsto \gamma_r^*$ as above:

$$(2.4) \quad \Theta_\Pi(\gamma_r) = e(\mathbf{G}_{F_r}) \Theta_{\Pi^*}(\gamma_r^*) = e(\mathbf{G}_{F_r}) \Theta_{\Pi^*}(\sigma^*(\gamma_r^*)) = \Theta_\Pi(\sigma(\gamma_r)),$$

in other words, it implies $\Pi \cong \Pi \circ \sigma$.

2.4.2. Proof of part (2). We can write Π^* as a normalized parabolic induction $\Pi^* = i_{P_r^*}^{G_r^*}(\pi_{M_r^*})$ for some F_r -rational parabolic subgroup with Levi decomposition $\mathbf{P}_r^* = \mathbf{M}_r^* \mathbf{N}_r^*$ and with F_r -points $P_r^* = M_r^* N_r^*$, and with $\pi_{M_r^*}$ a discrete series representation of M_r^* . If $\mathbf{M}_r^*$ transfers to an F_r -Levi subgroup $\mathbf{M}_r$ of $\mathbf{G}_{F_r}$ (a Levi factor of an F_r -parabolic $\mathbf{P}_r \subset \mathbf{G}_{F_r}$ with F_r -points P_r), then $\pi_{M_r^*}$ has a Langlands-Jacquet transfer π_{M_r} on M_r . The representation $i_{P_r}^{G_r}(\pi_{M_r})$ is irreducible and tempered by [Tad15, Cor. 6.3], and hence is a Langlands-Jacquet transfer of Π^* . So the only obstruction to transferring Π^* is the failure of $\mathbf{M}_r^*$ to transfer.

Moreover, note that a tempered irreducible $\pi^* = i_{P^*}^{G^*}(\sigma^*)$ has base-change lift of the form $\Pi^* = i_{P_r^*}^{G_r^*}(\pi_{M_r^*})$, where $\pi_{M_r^*}$ is the base-change lift of σ^* .

In light of the above, we need to prove the following twisted version of a result of Badulescu (see [Bad03, Lem. 3.3]). For H the rational points of p -adic group over a p -adic field, let $\Pi(H)$ denote the set of isomorphism classes of smooth irreducible representations of H .

Lemma 2.4.2. *Let $\psi \in \mathcal{H}(G^*)$. Suppose $O_{\gamma^*}(\psi) = 0$ for all semisimple elements $\gamma^* \in G^*$ which are not norms from G_r . Then $\text{Tr}(\psi|\pi^*) = 0$ for all $\pi^* \in \Pi(G^*)$ which are normalized parabolically induced representations $\pi^* = i_{P^*}^{G^*}(\sigma^*)$ where $P^* = M^*N^*$ is a standard parabolic subgroup of G^* such that $\mathbf{M}_{F_r}^* = \mathbf{M}^* \otimes_F F_r$ does not transfer to $\mathbf{G}_{F_r} = \mathbf{G} \otimes_F F_r$, and $\sigma^* \in \Pi(M^*)$.*

Proof. Let $K^* \subset G^*$ be a hyperspecial maximal compact subgroup in good position with respect to P^* , so that an Iwasawa decomposition holds: $G^* = M^*N^*K^*$. Then define the normalized constant term $\psi^{P^*} \in \mathcal{H}(M^*)$ by the formula

$$\psi^{P^*}(m^*) = \delta_{P^*}^{1/2}(m^*) \int_{N^*} \int_{K^*} \psi(k^{*-1}m^*n^*k^*) dk^* dn^*,$$

where dk^* and dn^* are Haar measures on K^* and N^* normalized by

$$\text{vol}_{dk^*}(K^*) = \text{vol}_{dn^*}(N^* \cap K^*) = 1.$$

Let $D_{G^*/M^*}(m^*)$ denote the normalized absolute value of $\det(1 - \text{Ad}(m^{*-1})| \text{Lie}(G^*)/\text{Lie}(M^*))$. Then

$$(2.5) \quad \text{Tr}(\psi | i_{P^*}^{G^*}(\sigma^*)) = \text{Tr}(\psi^{P^*} | \sigma^*)$$

$$(2.6) \quad O_{m^*}^{G^*}(\psi) = D_{G^*/M^*}^{-1/2}(m^*) O_{m^*}^{M^*}(\psi^{P^*}), \quad \forall m^* \in G^{*\text{rs}} \cap M^*$$

where $G^{*\text{rs}} \cap M^*$ is the set of elements of M^* which are regular semisimple in G^* (see e.g. [Lau96, Prop. 4.3.11; Lem. 7.5.7]).

If (2.5) is not zero, then applying the Weyl integration formula to ψ^{P^*} , we see that there must exist $m^* \in G^{*\text{rs}} \cap M^*$ such that $O_{m^*}(\psi^{P^*}) \neq 0$; hence by (2.6), $O_{m^*}^{G^*}(\psi) \neq 0$. Then by assumption m^* is a norm from G_r , say of $\delta \in G_r$.

Let $\eta : \mathbf{G} \rightarrow \mathbf{G}^*$ be a fixed inner twisting over F . There exists $h \in G^*(\bar{F})$ such that $h\eta(N_r\delta)h^{-1} = m^*$. By [Kot82, Lem. 5.8], “projection to the last coordinate” followed by $\text{Int}(h) \circ \eta$ induces an F -inner twisting $G_{\delta\sigma} \rightarrow G_{m^*}^*$, which is an F -isomorphism since both sides are F -tori. Further, over F_r , a similar argument shows that there is an F_r -isomorphism $G_{\delta\sigma, F_r} \cong G_{F_r, N_r\delta}$. Putting these together, there is an isomorphism of F_r -tori $\mathbf{G}_{m^*, F_r}^* \cong \mathbf{G}_{F_r, N_r\delta}$, which are maximal F_r -tori in $\mathbf{M}_{F_r}^*$ (resp., $\mathbf{G}_{F_r}$). Therefore, the F_r -split component $A_{\mathbf{M}_{F_r}^*}$ of the center of $\mathbf{M}_{F_r}^*$ transfers to $\mathbf{G}_{F_r}$, which means that $\mathbf{M}_{F_r}^*$ itself transfers to $\mathbf{G}_{F_r}$. The assertion is proved. $\square$

2.4.3. Remark on a relative version of (1). In various reductions coming into the proof of (1), we need to be able to change the group and the field of definition. In fact we will prove a more general version, where the base field $\mathbb{Q}_p$ is replaced by any finite extension E thereof. We can consider E -inner forms $\mathbf{G}, \mathbf{G}^*$, giving rising as above to inner forms $\mathbf{G}_{r'}$ and $\mathbf{G}_{r'}^*$ corresponding to a degree r' unramified extension field $E_{r'}$ of E . The statements are then the obvious analogues of (1) above, with σ replaced by any generator σ_E of $\text{Gal}(E_{r'}/E)$. Note that σ_E (not σ) is then used in the definition of twisted orbital integral and the notion of stable base change, when working relative to the extension $E_{r'}/E$. Although our proof will establish this more general version, to keep the notation simple we shall explain the proof over $E = \mathbb{Q}_p$, and after making the reductions in which $\mathbb{Q}_p$ is possibly replaced by a larger extension, we will then pretend that we are again working over $\mathbb{Q}_p$ and r is once again r' .

2.4.4. Reduction to a totally ramified Weil restriction. First we may assume that $\mathbf{G}^* = \text{Res}_{F/\mathbb{Q}_p} \text{GL}_n$ where F is a finite extension of $\mathbb{Q}_p$. By Shapiro’s Lemma, any $\mathbb{Q}_p$ -inner form of $\mathbf{G}^* = \text{Res}_{F/\mathbb{Q}_p} \text{GL}_n$ is of the form $\mathbf{G} = \text{Res}_{F/\mathbb{Q}_p} \mathbf{G}_1$ where $\mathbf{G}_1$ is an F -inner form of GL_n .

Let $F/E/\mathbb{Q}_p$ be the unique subextension with F/E totally ramified and $E/\mathbb{Q}_p$ unramified. For any field K with $\mathbb{Q}_p \subseteq K \subset \bar{\mathbb{Q}}_p$, let $\Gamma_K = \text{Gal}(\bar{\mathbb{Q}}_p/K)$ be its absolute Galois group, viewed as a

subgroup of $\Gamma_{\mathbb{Q}_p}$. By transitivity of Weil restriction, we have

$$\mathbf{G} = \text{Res}_{F/\mathbb{Q}_p} \mathbf{G}_1 = \text{Res}_{E/\mathbb{Q}_p} \left(\text{Res}_{F/E} \mathbf{G}_1 \right) =: \text{Res}_{E/\mathbb{Q}_p} (\mathbf{G}')$$

where this defines $\mathbf{G}'$. Let $\mathbf{G}'^* = \text{Res}_{F/E} \text{GL}_{n,F}$ be the quasi-split inner form of $\mathbf{G}'$ over E . Then $\mathbf{G}^* = \text{Res}_{E/\mathbb{Q}_p} \mathbf{G}'^*$ is the quasi-split inner form of $\mathbf{G}$.

Suppose $E = \mathbb{Q}_{p^s}$ is the unique unramified extension of $\mathbb{Q}_p$ of degree $s \geq 1$. Let $d := \gcd(r, s)$ and write $r = dr', s = ds'$. Let $E_{r'} = \mathbb{Q}_{p^{sr'}} = \mathbb{Q}_{p^{r's'}}$ be the unique degree r' unramified extension of $E = \mathbb{Q}_{p^s}$, which is also the unique degree s' unramified extension of $\mathbb{Q}_{p^r}$. Choose $a, b \in \mathbb{Z}$ such that $as + br = d$, or in other words $as' + br' = 1$.

Lemma 2.4.3. *There exists an isomorphism of algebras $E \otimes_{\mathbb{Q}_p} \mathbb{Q}_{p^r} \cong (E_{r'})^d$ under which $1 \otimes \sigma$ corresponds to*

$$(x_1, \dots, x_d) \mapsto (x_2, \dots, x_d, \tau(x_1))$$

where $\tau = \sigma^d = (\sigma^s)^a$.

Proof. There is an obvious isomorphism $\mathbb{Q}_{p^s} \otimes_{\mathbb{Q}_p} \mathbb{Q}_{p^r} \cong \mathbb{Q}_{p^s} \otimes_{\mathbb{Q}_{p^d}} (\mathbb{Q}_{p^d} \otimes_{\mathbb{Q}_p} \mathbb{Q}_{p^r})$. There is an isomorphism

$$(2.7) \quad \begin{aligned} \mathbb{Q}_{p^d} \otimes_{\mathbb{Q}_p} \mathbb{Q}_{p^r} &\xrightarrow{\sim} (\mathbb{Q}_{p^r})^d \\ x \otimes y &\mapsto (\sigma^{-i+1}(x) y)_{i=1}^d \end{aligned}$$

under which $\text{id} \otimes \sigma$ goes over to the automorphism

$$(x_1, x_2, \dots, x_d) \mapsto (\sigma(x_2), \sigma(x_3), \dots, \sigma(x_d), \sigma(x_1)).$$

If we post-compose (2.7) with the automorphism of the algebra $(\mathbb{Q}_{p^r})^d$ given by

$$(x_1, x_2, \dots, x_d) \mapsto (x_1, \sigma(x_2), \dots, \sigma^{d-1}(x_d)),$$

then $\text{id} \otimes \sigma$ on $\mathbb{Q}_{p^d} \otimes_{\mathbb{Q}_p} \mathbb{Q}_{p^r}$ is identified with the automorphism of $(\mathbb{Q}_{p^r})^d$ given by

$$(x_1, x_2, \dots, x_d) \mapsto (x_2, x_3, \dots, \tau(x_1)).$$

This map is now $\mathbb{Q}_{p^d}$ -linear and furthermore since $\sigma^r = \text{id}$, we see $\sigma^d = (\sigma^s)^a$. Therefore $\text{id} \otimes \sigma$ on

$$E \otimes_{\mathbb{Q}_p} \mathbb{Q}_{p^r} = \mathbb{Q}_{p^s} \otimes_{\mathbb{Q}_{p^d}} (\mathbb{Q}_{p^r})^d = (\mathbb{Q}_{p^{s'r'd}})^d$$

is given by the same formula. $\square$

We will view $\tau := \sigma^d = (\sigma^s)^a$ as a generator of $\text{Gal}(E_{r'}/E)$ (note that since σ^s is a generator and a is relatively prime to r' , τ is also a generator).

This leads to the decompositions

$$\mathbf{G}_r := \text{Res}_{\mathbb{Q}_{p^r}/\mathbb{Q}_p} \mathbf{G}_{\mathbb{Q}_{p^r}} \cong (\text{Res}_{E/\mathbb{Q}_p} \mathbf{G}'_{r'})^d, \quad \mathbf{G}_r^* \cong (\text{Res}_{E/\mathbb{Q}_p} \mathbf{G}'_{r'}^*)^d$$

where $\mathbf{G}'_{r'} := \text{Res}_{E_{r'}/E} (\mathbf{G}' \otimes_E E_{r'})$ and $\mathbf{G}'_{r'}^* := \text{Res}_{E_{r'}/E} (\mathbf{G}'^* \otimes_E E_{r'})$. The σ -actions on the right hand sides are given by the same formula as above.

The σ -stable representation Π^* of $\mathbf{G}_r^* = \mathbf{G}_r^*(\mathbb{Q}_p)$ is decomposed into $\Pi^* = (\Pi'^*)^{\otimes d}$, where Π'^* is a τ -stable representation of $\mathbf{G}'_{r'}^*(E) = \mathbf{G}'_{r'}^*(E_{r'})$. Let I_τ^* be the canonical intertwining operator on Π'^* . Then the canonical intertwining operator I_σ^* for Π^* is given by

$$I_\sigma^*(v_1 \otimes \dots \otimes v_d) = v_2 \otimes \dots \otimes v_d \otimes I_\tau^* v_1, \quad \forall v_1, \dots, v_d \in \Pi'.$$

For any function $\phi = \phi_1 \otimes \dots \otimes \phi_d \in C_c^\infty(G_r^*) = C_c^\infty(\mathbf{G}'_{r'}^*(E_{r'}))^{\otimes d}$, let $\phi' := \phi_1 * \dots * \phi_d$ be the convolution product of the factors. Then by [AC89, §I.5], the functions ϕ and ϕ' have matching twisted orbital integrals and hence they have the same base change transfer $\phi^* \in C_c^\infty(G^*)$. On the other hand, by the Saito-Shintani formula (see for example [Fen20, Lemma 6.12]) we deduce that

$$\text{Tr}(\phi I_\sigma^* | \Pi^*) = \text{Tr}(\phi' I_\tau^* | \Pi'^*).$$

Indeed, to show this we are reduced to the case where for each $1 \leq i \leq d$, ϕ_i is the characteristic function of a coset $g_i K$ for $g_i \in \mathbf{G}'_{r'}^*(E_{r'})$ where $K \subset \mathbf{G}'_{r'}^*(E_{r'})$ is a normal open compact subgroup.

Let V^K be the space of K -invariants in Π'^* . Then after dividing the volume factor $\text{vol}(K)^d$ the identity above becomes

$$\text{Tr}((g_1 \otimes \cdots \otimes g_d)I_\sigma^* | (V^K)^{\otimes d}) = \text{Tr}(g_1 \cdots g_d I_\tau^* | V^K)$$

and this follows from [Fen20, Lemma 6.12].

In particular, we see that Π'^* is the base change lift of π^* with respect to the extension $E_{r'}/E$. Moreover Π^* has a Langlands-Jacquet transfer to G_r if and only if Π'^* has a Langlands-Jacquet transfer to $\mathbf{G}'_{r'}(E) = \mathbf{G}'(E_{r'})$ by part (2), which we already proved above.

Now suppose that the theorem is true for the data $(\mathbf{G}'_{r'}(E), \tau \in \text{Gal}(E_{r'}/E))$.

Suppose Π^* has a Langlands-Jacquet transfer to G_r . Then Π'^* has a Langlands-Jacquet transfer Π' , which is a representation of $\mathbf{G}'_{r'}(E) = \mathbf{G}'(E_{r'})$. By the validity of part (1) for the data $(\mathbf{G}'_{r'}(E), \tau \in \text{Gal}(E_{r'}/E))$, the representation Π' is τ -stable and there is an intertwining operator I_τ such that

$$\text{Tr}(\phi' I_\tau | \Pi') = e(\mathbf{G}'_{r'}) \text{Tr}(\phi^* | \pi^*)$$

for all $\phi' \in C_c^\infty(\mathbf{G}'(E_{r'}))$ with stable base change transfer $\phi^* \in C_c^\infty(G^*)$. Let $\Pi := (\Pi')^{\otimes d}$ be the representation of $G_r = \mathbf{G}'_{r'}(E)^d$ and define an endomorphism I_σ on Π by the formula

$$I_\sigma(v_1 \otimes \cdots \otimes v_d) = v_2 \otimes \cdots \otimes v_d \otimes I_\tau v_1.$$

Then one checks that Π is σ -stable and I_σ is an intertwining operator. Since Π' is the Langlands-Jacquet transfer of Π'^* , it follows that $\Pi = (\Pi')^{\otimes d}$ is the Langlands-Jacquet transfer of $\Pi^* = (\Pi'^*)^{\otimes d}$. For any $\phi = \phi_1 \otimes \cdots \otimes \phi_d \in C_c^\infty(G_r)$ with $\phi' := \phi_1 * \cdots * \phi_d$, by the Saito-Shintani formula again we get

$$\text{Tr}(\phi I_\sigma | \Pi) = \text{Tr}(\phi' I_\tau | \Pi') = e(\mathbf{G}'_{r'}) \text{Tr}(\phi^* | \pi^*)$$

where in the first equality we use that ϕ and ϕ' have the same stable base change transfer ϕ^* (by the same reasoning as in [AC89, §1.5]).

From [Kot83, Corollary on page 295] we have the following identities of Kottwitz signs:

$$\begin{aligned} e(\mathbf{G}_r) &= e(\mathbf{G})^r = e(\mathbf{G}')^r \\ e(\mathbf{G}'_{r'}) &= e(\mathbf{G}')^{r'}. \end{aligned}$$

In other words, we have $e(\mathbf{G}_r) = e(\mathbf{G}'_{r'})^d$. In case $e(\mathbf{G}_r) \neq e(\mathbf{G}'_{r'})$, we must have $e(\mathbf{G}'_{r'}) = -1$ and d is even. Then $r = r'd$ is also even and we can replace I_σ by $-I_\sigma$ to get

$$\text{Tr}(\phi I_\sigma | \Pi) = e(\mathbf{G}_r) \text{Tr}(\phi^* | \pi^*)$$

and hence part (1) for the data $(G_r, \sigma \in \text{Gal}(\mathbb{Q}_{p^r}/\mathbb{Q}_p))$ follows.

We have now reduced the proof of the Theorem for the data $(\mathbf{G}_r(\mathbb{Q}_p), \sigma \in \text{Gal}(\mathbb{Q}_{p^r}/\mathbb{Q}_p))$ to the proof of the Theorem for the data $(\mathbf{G}'_{r'}(E), \tau \in \text{Gal}(E_{r'}/E))$. In this way we reduced from the group $\mathbf{G}$ over $\mathbb{Q}_p$ to the group $\mathbf{G}'$ over E , which is a Weil-restriction for a *totally ramified* extension F/E . So we now abuse notation and pretend we are in the situation where $\mathbf{G} = \text{Res}_{F/\mathbb{Q}_p} \mathbf{G}_1$ and $F/\mathbb{Q}_p$ is *totally ramified*. The arguments below work the same way in the actual situation where we would be working over the base field E instead of $\mathbb{Q}_p$.

2.4.5. Analysis of $F/\mathbb{Q}_p$ totally ramified case. In view of our reduction above to the case where $F/\mathbb{Q}_p$ is totally ramified, we may assume $F \otimes_{\mathbb{Q}_p} \mathbb{Q}_{p^r} = F_r$ is the unique degree r unramified extension of F . For the rest of this section we fix the following notations. Let $B = M_s(D)$ be a central simple algebra over F of dimension n^2 , where D is a division algebra over F of dimension d^2 , so that $sd = n$. Let $B_r := B \otimes_F F_r$ and write $B_r = M_t(\tilde{D})$ where $\tilde{D}$ is a division algebra over F_r of dimension $\tilde{d}^2$ so that $t\tilde{d} = n$ and $\tilde{d}$ divides d . Then $G = B^\times$ and $G_r = B_r^\times$ are the unit groups. More precisely, we view $B^\times$ (resp., $B_r^\times$) via Weil restriction as an algebraic group over $\mathbb{Q}_p$ (resp., $\mathbb{Q}_{p^r}$).

The proof of (1) will occupy the rest of this subsection. It will involve a global method and will rely on a comparison of twisted trace formulas. We first observe that in the current situation the sign in part (1) is

$$e(\mathbf{G}_r) = (-1)^{n-t}.$$

Indeed, this follows from part (6) of the Corollary on p. 295 of [Kot83] and the discussion at the bottom of p. 296 of *loc. cit.* Thus the identity to be proved becomes

$$\mathrm{Tr}(\phi I_\sigma | \pi) = (-1)^{n-t} \mathrm{Tr}(\phi^* | \pi^*)$$

For simplicity, in the proof we will view G and G_r as algebraic groups over F without further comment.

2.4.6. Remark on central characters. Our next task is to introduce the global method, relying on the Deligne-Kazhdan very simple trace formula. A technical point is that this formula (and its twisted version) is valid only for automorphic forms which transform by a unitary central character under translation of the argument by the center. The groups we work with are not adjoint, but have split connected centers. Theorem 2.4.1 concerns tempered representations and their base changes. Tempered representations have a unitary central character, which we regard as fixed throughout the discussion, but we suppress from our notation. We will assume these conventions for the rest of this section without further comment. For example, for any nonarchimedean local field F , we write $\mathcal{H}(\mathbb{G}(F))$ for the Hecke algebra of smooth functions on $\mathbb{G}(F)$ having compact modulo center support, and which transform by the inverse of the aforementioned unitary character of the center.

2.4.7. Approximation by automorphic representations. Let $\tilde{\mathbb{F}}/\mathbb{F}$ be a cyclic degree r extension of number fields and v_0 a place of $\mathbb{F}$ such that

- v_0 is inert in $\tilde{\mathbb{F}}$ and the completion $\tilde{\mathbb{F}}_{v_0}/\mathbb{F}_{v_0}$ is isomorphic to F_r/F ;
- All archimedean places of $\mathbb{F}$ are complex.

We can choose a central division algebra $\mathbb{D}$ over $\mathbb{F}$ such that

- $\mathbb{D}_{v_0} \cong B$;
- Let $S(\mathbb{D}) = \{v_0, v_1, \dots, v_m\}$ be the set of places where $\mathbb{D}$ is ramified. We may arrange that for all $1 \leq i \leq m$, the place v_i splits completely in $\tilde{\mathbb{F}}$ and $\mathbb{D}_{v_i}$ is a division algebra.

The existence of $\mathbb{D}$ follows from the local-global exact sequence for Brauer groups in global class field theory, see for example [Mum08, §21, p.196].

Denote $\tilde{\mathbb{D}} := \mathbb{D} \otimes_{\mathbb{F}} \tilde{\mathbb{F}}$. Then by the second assumption above, $\tilde{\mathbb{D}}$ is a central division algebra over $\tilde{\mathbb{F}}$. Let $\mathbb{G} := \mathbb{D}^\times$ be the unit group of $\mathbb{D}$ and $\tilde{\mathbb{G}} := \mathrm{Res}_{\tilde{\mathbb{F}}/\mathbb{F}} \tilde{\mathbb{D}}^\times$. Then $\mathbb{G}$ is an inner form of $\mathbb{G}^* := \mathrm{GL}_n$ and $\tilde{\mathbb{G}}$ is an inner form of $\tilde{\mathbb{G}}^* := \mathrm{Res}_{\tilde{\mathbb{F}}/\mathbb{F}} \mathrm{GL}_n$. We fix an isomorphism $\tilde{\mathbb{G}}_{v_0} \cong G_r$.

Moreover we choose two non-archimedean places v_{m+1}, v_{m+2} of $\mathbb{F}$ not in the set $\{v_0, \dots, v_m\}$ such that v_{m+1} splits completely in $\tilde{\mathbb{F}}$ and v_{m+2} is inert in $\tilde{\mathbb{F}}$.

Fix a supercuspidal representation ρ of $\mathrm{GL}_n(\mathbb{F}_{v_{m+1}})$. For later purposes, choose a matrix coefficient f_ρ of ρ and an idempotent $e_\rho \in \mathcal{H}(\mathrm{GL}_n(\mathbb{F}_{v_{m+1}}))$ such that $f_\rho * e_\rho = f_\rho$.

Let $\mathcal{A}^*$ be the set of σ -stable cuspidal automorphic representations $\tilde{\Pi}^*$ of $\tilde{\mathbb{G}}^*(\mathbb{A}_{\mathbb{F}}) = \mathrm{GL}_n(\mathbb{A}_{\tilde{\mathbb{F}}})$ satisfying the following conditions:

- $\tilde{\Pi}_{v_0}^*$ is d -compatible in the sense of [Bad08]: there exists a regular semisimple element $\gamma^* \in \tilde{\mathbb{G}}^*(\mathbb{F}_{v_0})$ which is a transfer of an element in $\tilde{\mathbb{G}}(\mathbb{F}_{v_0})$, such that $\Theta_{\tilde{\Pi}_{v_0}^*}(\gamma^*) \neq 0$;
- For all $1 \leq i \leq m$, at any place w of $\tilde{\mathbb{F}}$ above v_i , $\tilde{\Pi}_w^*$ is isomorphic to the Steinberg representation;
- At any place w above v_{m+1} , $\tilde{\Pi}_w^*$ is isomorphic to the supercuspidal representation ρ fixed above.
- At the unique place of $\tilde{\mathbb{F}}$ above v_{m+2} , $\tilde{\Pi}^*$ is isomorphic to the Steinberg representation.

Note that, like any local component of a cuspidal automorphic representation, $\tilde{\Pi}_{v_0}^*$ is unitary and generic. We remark here that since $\tilde{\Pi}_{v_0}^*$ is generic and σ -stable, it is the base-change of a generic representation of G^* , characterized by Harish-Chandra character identities (see [AC89, p. 59-60] and [Zel80, Thm. 9.7]). We will see in Proposition 2.4.8 below that the set $\mathcal{A}^*$ is nonempty.

By [Bad08, Thm. 5.1 and Prop. 5.5], such a $\tilde{\Pi}^*$ has a global Langlands-Jacquet transfer to a discrete automorphic representation $\tilde{\Pi}$ of $\tilde{\mathbb{G}}$ and moreover $\tilde{\Pi}$ is cuspidal. Let $\mathcal{A}$ be the set of

automorphic representations of $\tilde{\mathbb{G}}$ that are global Langlands-Jacquet transfers (in Badulescu's sense) of representations in $\mathcal{A}^*$.

Lemma 2.4.4. *The set $\mathcal{A}$ consists of σ -stable cuspidal automorphic representations $\tilde{\Pi}$ of $\tilde{\mathbb{G}}(\mathbb{A}_{\mathbb{F}})$ such that*

- For all $1 \leq i \leq m$, at any place w of $\tilde{\mathbb{F}}$ above v_i , $\tilde{\Pi}_w$ is the trivial representation;
- At any place w above v_{m+1} , $\tilde{\Pi}_w$ is isomorphic to the supercuspidal representation ρ .
- At the unique place of $\tilde{\mathbb{F}}$ above v_{m+2} , $\tilde{\Pi}$ is isomorphic to the Steinberg representation.

Proof. Let $\tilde{\Pi} \in \mathcal{A}$ be the global Jacquet-Langlands transfer of $\tilde{\Pi}^* \in \mathcal{A}^*$. Then clearly $\tilde{\Pi}$ satisfies the stated conditions at the places w above the places v_i , $1 \leq i \leq m+1$. The fact that $\tilde{\Pi} \in \mathcal{A}$ is σ -stable follows using Remark 2.4.1 along with Flath's article [Fla79]. In the other direction, [Bad08, Appendix] shows that every cuspidal representation $\tilde{\Pi}$ arises as a Badulescu-Langlands-Jacquet transfer of a cuspidal representation $\tilde{\Pi}^*$ whose local factors are all d -compatible. If the local factors of $\tilde{\Pi}$ satisfy the properties above, then $\tilde{\Pi}^*$ must satisfy the four properties defining $\mathcal{A}^*$. Further, the σ -invariance of $\tilde{\Pi}$ forces, by injectivity of Badulescu's map $\mathbf{G}^{-1}$ in [Bad08, Prop. 5.5], that $\tilde{\Pi}^*$ is also σ -invariant. Hence $\tilde{\Pi}$ belongs to $\mathcal{A}$. $\square$

Remark 2.4.5. Note we did not impose a condition at v_0 in Lemma 2.4.4. In fact $\tilde{\Pi} \in \mathcal{A}$ forces $\tilde{\Pi}_{v_0}$ to be an irreducible unitary representation which is in the image of a d -compatible representation of $\tilde{\mathbb{G}}^*(\mathbb{F}_{v_0})$ under the map $|\mathrm{LJ}^u|_{v_0}$ of [Bad08, Prop. 3.10]. But not every irreducible generic unitary representation of $\tilde{\mathbb{G}}(\mathbb{F}_{v_0})$ is in the image of $|\mathrm{LJ}^u|_{v_0}$; see [Bad08, Lem. 3.11]. Hence this property of $\tilde{\Pi}_{v_0}$ is highly non-obvious, and ultimately follows from the results of [Bad08, Appendix], by the argument above.

Lemma 2.4.6. *Let $\tilde{\Pi}^*$ be a σ -stable cuspidal automorphic representation of $\tilde{\mathbb{G}}^*$ in the set $\mathcal{A}^*$. Let $\pi_{v_0}^*$ be an irreducible generic smooth representation of $\mathbb{G}^*(\mathbb{F}_{v_0})$ whose base change to $\mathbb{G}^*(\tilde{\mathbb{F}}_{v_0}) = \tilde{\mathbb{G}}^*(\mathbb{F}_{v_0})$ is $\tilde{\Pi}_{v_0}^*$. Let $\tilde{\Pi} \in \mathcal{A}$ be the Badulescu-Langlands-Jacquet transfer of $\tilde{\Pi}^* \in \mathcal{A}^*$. Then we can normalize the intertwining operator I_{σ} on the σ -stable representation $\tilde{\Pi}_{v_0}$ of $\tilde{\mathbb{G}}(\mathbb{F}_{v_0}) = \mathbb{G}(\tilde{\mathbb{F}}_{v_0})$ such that*

$$\mathrm{Tr}(\phi_{v_0} I_{\sigma} | \tilde{\Pi}_{v_0}) = e(\mathbf{G}_r) \mathrm{Tr}(\phi_{v_0}^* | \pi_{v_0}^*).$$

for every $\phi_{v_0} \in \mathcal{H}(\tilde{\mathbb{G}}(\mathbb{F}_{v_0}))$ whose stable base change transfer is $\phi_{v_0}^* \in \mathcal{H}(\mathbb{G}^*(\mathbb{F}_{v_0}))$.

Proof. We consider test functions $\phi = \otimes_w \phi_w \in \mathcal{H}(\tilde{\mathbb{G}}(\mathbb{A}_{\mathbb{F}})) = \mathcal{H}(\mathbb{G}(\mathbb{A}_{\tilde{\mathbb{F}}}))$ that satisfy the following conditions (here w runs over all places of $\tilde{\mathbb{F}}$):

- For any $1 \leq i \leq m$, fix one place w_i of $\tilde{\mathbb{F}}$ above v_i . For any place $w|v_i$ different from w_i , $\phi_w = e_{v_i}$ is an idempotent such that $\phi_{w_i} * e_{v_i} = \phi_{w_i}$, where e_{v_i} and ϕ_{w_i} are understood as functions on $\mathbb{G}(\mathbb{F}_{v_i})$ under fixed isomorphisms $\mathbb{F}_{v_i} \cong \tilde{\mathbb{F}}_w \cong \tilde{\mathbb{F}}_{w_i}$. We assume that the support of ϕ_{w_i} is contained in the elliptic regular semisimple locus of $\mathbb{G}(\mathbb{F}_{v_i})$.
- Fix a place w_{m+1} above v_{m+1} , we require $\phi_{w_{m+1}}$ to be a matrix coefficient of the supercuspidal representation $\rho \cong \tilde{\Pi}_{w_{m+1}} \cong \tilde{\Pi}_{w_{m+1}}^*$ and for any $w|v_{m+1}$ different from w_{m+1} , ϕ_w is the idempotent e_{ρ} so that $\phi_{w_{m+1}} * e_{\rho} = \phi_{w_{m+1}}$.
- At the unique place w_{m+2} of $\tilde{\mathbb{F}}$ above v_{m+2} , we require $\phi_{w_{m+2}}$ to be a pseudo-coefficient of the Steinberg representation of $\tilde{\mathbb{G}}(\mathbb{F}_{v_{m+2}}) = \mathbb{G}(\tilde{\mathbb{F}}_{w_{m+2}})$.
- For any non-archimedean place $v \notin \{v_i\}_{0 \leq i \leq m+2}$, ϕ_v is in the unramified Hecke algebra and is the unit element for all but finitely many such v .

For each place v of $\mathbb{F}$, let $\phi_v^* \in \mathcal{H}(\mathbb{G}^*(\mathbb{F}_v))$ be the stable base change transfer of $\otimes_{w|v} \phi_w$ and let $\phi^* := \otimes_v \phi_v^*$. (In particular, at $v = v_0$, the $\phi_{v_0}^*$ and ϕ_{v_0} range freely over all matching pairs of functions.) Then we can apply the simple twisted trace formula to the test functions ϕ, ϕ^* . Comparison of the geometric sides leads to the identity

$$(2.8) \quad r \mathrm{Tr}(R_{\tilde{\mathbb{G}}, \mathrm{cusp}}(\phi) I_{\sigma}) = \mathrm{Tr}(R_{\mathbb{G}^*, \mathrm{cusp}}(\phi^*))$$

for all pairs ϕ, ϕ^* we consider. Here I_σ on the left hand side refers to the canonical intertwining operator on L^2 -functions given by the action of σ on the argument. To prove (2.8) we follow the reasoning of [AC89, p.44], making use of Lemma 2.1.1. Then following the arguments in [AC89, bottom of page 55], we deduce¹ that there is a cuspidal automorphic representation π^* of $\mathbb{G}^*$ satisfying

$$(2.9) \quad \mathrm{Tr}(\phi I_\sigma | \tilde{\Pi}) = \mathrm{Tr}(\phi^* | \pi^*).$$

The argument of [AC89] goes over to our situation without change, except that we need to invoke the following lemma (the analogue of which appears in [AC89, top of p. 56]).

Lemma 2.4.7. *Suppose ρ and ρ' are two smooth irreducible representations of $\mathbb{G}^*(\mathbb{F}_{v_i})$. If $f^* \in \mathcal{H}(\mathbb{G}^*(\mathbb{F}_{v_i}))$ is in the image of the stable transfer map from $\tilde{\mathbb{G}}$, and if $\rho' = \rho \otimes \chi_{v_i}$ for some character χ of $\mathbb{F}^\times N_r(\mathbb{A}_{\mathbb{F}}^\times) \backslash \mathbb{A}_{\mathbb{F}}^\times$, then*

$$\mathrm{Tr}(f^* | \rho) = \mathrm{Tr}(f^* | \rho').$$

Proof. In the Weyl integration formula, over each $\mathbb{F}_{v_i}$ -rational torus T^* in $\mathbb{G}_{\mathbb{F}_{v_i}}^*$ representing a conjugacy class of such tori, one integrates $\Theta_\rho(t^*) O_{t^*}(f^*) dt^*$ over $t^* \in T^*(\mathbb{F}_{v_i})$, times some normalizing factors. This only contributes if $t^* = \mathcal{N}_r(\delta)$ for some $\delta \in \tilde{\mathbb{G}}(\mathbb{F}_{v_i})$. Then we use that

$$\Theta_\rho(\mathcal{N}_r(\delta)) = \Theta_{\rho'}(\mathcal{N}_r(\delta)),$$

which holds because χ_{v_i} is identically 1 on elements of the form $\det(N_r(\delta))$, as the latter can be viewed as $\mathbb{F}_{v_i}$ -points of the $\mathbb{F}_{v_i}$ -isomorphic abelianized $\mathbb{F}_{v_i}$ -groups $\tilde{\mathbb{G}}_{\mathbb{F}_{v_i}}^{\mathrm{ab}} \cong [\tilde{\mathbb{G}}_{\mathbb{F}_{v_i}}^*]^{\mathrm{ab}}$. $\square$

For each $1 \leq i \leq m$, we fix the local intertwining operator I_{σ, v_i} on $\tilde{\Pi}_{v_i} \cong \tilde{\Pi}_{w_i}^{\otimes r}$ to be given by the cyclic permutation $I_{\sigma, v_i}(x_1 \otimes \cdots \otimes x_r) = x_2 \otimes x_3 \otimes \cdots \otimes x_r \otimes x_1$. Let $\phi_{w_i}^*$ be a Jacquet-Langlands transfer of ϕ_{w_i} . Then by Lemma 2.3.2(3) – rather by its trivial central character variant, see Remark 2.3.2 – $\phi_{w_i}^*$ is a stable base change for the local test function $\phi_{v_i} = \phi_{w_i} \otimes e_{v_i} \otimes \cdots \otimes e_{v_i}$. Moreover, we have

$$\mathrm{Tr}(\phi_{v_i} I_{\sigma, v_i} | \tilde{\Pi}_{v_i}) = \mathrm{Tr}(\phi_{w_i} | \tilde{\Pi}_{w_i}) = (-1)^{n-1} \mathrm{Tr}(\phi_{w_i}^* | \pi_{v_i}^*).$$

The first equality follows from the Saito-Shintani formula. For the second equality we have used that $\tilde{\Pi}$ is the Badulescu-Langlands-Jacquet transfer of a unique $\tilde{\Pi}^* \in \mathcal{A}^*$, and by construction $\pi_{v_i}^* \cong \tilde{\Pi}_{w_i}^*$ is the local Jacquet-Langlands transfer of $\tilde{\Pi}_{w_i}$.

For any place v of $\mathbb{F}$ not in the set $\{v_i, 0 \leq i \leq m\}$, we have $\tilde{\mathbb{G}}(\mathbb{F}_v) \cong \mathrm{GL}_n(\tilde{\mathbb{F}} \otimes_{\mathbb{F}} \mathbb{F}_v)$ and we fix the intertwining operator $I_{\sigma, v}$ on $\tilde{\Pi}_v$ as in [AC89]. We claim that at the place v_{m+2} we have

$$\mathrm{Tr}(\phi_{w_{m+2}} I_{\sigma, w_{m+2}} | \tilde{\Pi}_{w_{m+2}}) = \mathrm{Tr}(\phi_{v_{m+2}}^* | \pi_{v_{m+2}}^*)$$

where $\tilde{\Pi}_{w_{m+2}}$ is isomorphic to the Steinberg representation of $\mathbb{G}(\tilde{\mathbb{F}}_{w_{m+2}}) \cong \mathrm{GL}_n(\tilde{\mathbb{F}}_{w_{m+2}})$ by assumption. By our assumption on $\phi_{w_{m+2}}$, its stable base change transfer $\phi_{v_{m+2}}^*$ is a pseudo-coefficient of the Steinberg representation of $\mathbb{G}(\mathbb{F}_{v_{m+2}}) = \mathrm{GL}_n(\mathbb{F}_{v_{m+2}})$. Therefore as the factor $\mathrm{Tr}(\phi_{v_{m+2}}^* | \pi_{v_{m+2}}^*)$ appearing in (2.9) is not zero, we must have that $\pi_{v_{m+2}}^*$ is the Steinberg representation. Then the Steinberg representation $\tilde{\Pi}_{v_{m+2}}$ is its base-change lift, and we get the identity above.

On the other hand, at any place v of $\mathbb{F}$ not in $\{v_i\}_{1 \leq i \leq m+2}$ such that $\tilde{\Pi}_v$ is unramified, $I_{\sigma, v}$ is the identity on the one dimensional space of spherical vectors $\tilde{\Pi}_v^{\tilde{\mathbb{G}}(\mathcal{O}_{\mathbb{F}, v})}$. These choices determine uniquely an intertwining operator I_{σ, v_0} on $\tilde{\Pi}_{v_0}$ so that

$$\mathrm{Tr}(\phi_{v_0} I_{\sigma, v_0} | \tilde{\Pi}_{v_0}) = (-1)^{m(n-1)} \mathrm{Tr}(\phi_{v_0}^* | (\pi^*)_{v_0}),$$

where $(\pi^*)_{v_0}$ is the v_0 -component of π^* , which despite our notation is *a priori* different from the representation $\pi_{v_0}^*$ we started with.

¹It is for this step that we need to use the inert place v_{m+2} .

Next we show that the sign $(-1)^{m(n-1)}$ differs from the desired sign $e(\mathbf{G}_r) = (-1)^{n-t}$ by an r -th root of unity. Recall the notation from §2.4.7. In particular we have $n = td$. First suppose that n is odd, so t is also odd. Then $m(n-1) - (n-t)$ is even and we are done.

We assume from now on that n is even. Recall that $\mathbb{D}_{v_0} \cong \widetilde{\mathbb{D}}_{v_0} \cong B$. Write its invariant as $\text{inv}(B) = \frac{a}{d} = \frac{at}{n}$ where a is coprime to $\tilde{d} = n/t$. For each $1 \leq i \leq m$, write $\text{inv}(\mathbb{D}_{v_i}) = \frac{a_i}{n}$ where a_i is coprime to n . In particular, a_i is odd for all $1 \leq i \leq m$. The sum of local invariants of $\widetilde{\mathbb{D}}$ vanishes, which implies that

$$at + r(a_1 + \cdots + a_m) \equiv 0 \pmod{n}.$$

Suppose t is even, then we get that rm is even. If r is odd then m must be even and hence $m(n-1) - (n-t)$ is even. If r is even, then -1 is an r -th root of unity and we are done.

Finally if t is odd, then $\tilde{d}$ is even and a is odd (since it is coprime to $\tilde{d}$). Then the congruence above implies that rm is odd, so in particular m is odd. Then $m(n-1) - (n-t)$ is even and we are done.

Consequently after multiplying by an r -th root of unity if necessary, we get an intertwining operator I_{σ, v_0} on $\widetilde{\Pi}_{v_0}$ such that

$$(2.10) \quad \text{Tr}(\phi_{v_0} I_{\sigma, v_0} | \widetilde{\Pi}_{v_0}) = e(\mathbf{G}_r) \text{Tr}(\phi_{v_0}^* | (\pi^*)_{v_0}).$$

We regard (2.10) as equation in all matching pairs $\phi_{v_0}, \phi_{v_0}^*$. Using a similar comparison of simple (twisted) trace formulas, now for $\widetilde{\mathbb{G}}^*$ instead of $\widetilde{\mathbb{G}}$, we get an automorphic representation $\pi^\bullet$ of $\mathbb{G}^*$ whose global base change is $\widetilde{\Pi}^*$, and such that $\pi_{v_0}^\bullet \cong \pi_{v_0}^*$. Using [AC89, p.54, Theorem III.3.1], one shows that the automorphic representation π^* and $\pi^\bullet$ of $\mathbb{G}^*$ are isomorphic up to a twist by an adelic character χ as in Lemma 2.4.7. Indeed, we know that $\widetilde{\Pi}^* = \text{BC}(\pi^\bullet)$ by construction, that π^* is related to $\widetilde{\Pi}$ by (2.9), and that $\widetilde{\Pi}$ is the Badulescu-Langlands-Jacquet (BLJ) transfer of $\widetilde{\Pi}^*$. By the global-local compatibility of the BLJ correspondence and the fact that at almost all places w of $\mathbb{F}$, the local correspondence $\widetilde{\Pi}_w^* \mapsto \widetilde{\Pi}_w$ is induced by the isomorphism of the local groups, the relations prove that almost all places v of $\mathbb{F}$, $\text{BC}((\pi^*)_v) \cong \widetilde{\Pi}_v^*$. Thus by *loc. cit.*, $\pi^\bullet = \pi^* \otimes \chi$, for some character χ as in Lemma 2.4.7. Therefore, $\pi_{v_0}^*$ and $(\pi^*)_{v_0}$ differ by a character of $\mathbb{F}_{v_0}^\times$. Using that v_0 is inert in $\mathbb{F}$, we replace π^* by its twist by the corresponding character of $\mathbb{A}_{\mathbb{F}}^\times$. By the fact that every component of the global ϕ^* is in the image of the stable base-change map by construction, this twist does not affect the validity of (2.9), thanks to Lemma 2.4.7. We thereby arrange that the v_0 -component of π^* is exactly the given $\pi_{v_0}^*$, and the Lemma is proved. $\square$

Let $\mathcal{A}_{v_0}^*$ (resp. $\mathcal{A}_{v_0}$) be the subset of $\text{Irr}(G_r^*)$ (resp. $\text{Irr}(G_r)$) consisting of v_0 components of the automorphic representations in $\mathcal{A}^*$ (resp. $\mathcal{A}$).

Let $\mathfrak{S}_2(G_r)$ be the set of equivalence classes of pairs (L, D) where L is a Levi subgroup of $\mathbf{G}_{F_r}$ defined over F_r and D is an $X_{ur}^*(L)$ -orbit of a discrete series representation of L . For each $(L, D) \in \mathfrak{S}_2(G_r)$, let $V(L, D)$ be the corresponding component in the L^2 -Bernstein variety of G_r . By the local Jacquet-Langlands correspondence, we also view $V(L, D)$ as a component in the L^2 -Bernstein variety of G_r^* . We refer to [Shi12, §2] for more details.

Proposition 2.4.8. *For each σ -stable pair $(L, D) \in \mathfrak{S}_2(G_r^*)$, the set $\mathcal{A}_{v_0}^* \cap V(L, D)$ is Zariski dense in $V(L, D)$.*

Proof. Suppose on the contrary that the intersection is not Zariski dense. Then by the twisted version of the trace Paley-Wiener theorem [Rog88] (see also [AC89, Prop. 2.9, Cor. 2.10]), there exists a function $f_0 \in \mathcal{H}(G_r^*)$ whose twisted character vanishes on $\mathcal{A}_{v_0}^* \cap V(L, D)$ and all other Bernstein components, but does not vanish identically on $V(L, D)$. In particular, there exists $\delta_0 \in \widetilde{\mathbb{G}}^*(\mathbb{F}_{v_0})$ such that

$$TO_{\delta_0 \sigma}(f_0) \neq 0$$

We consider test functions $\phi = \otimes_w \phi_w \in \mathcal{H}(\widetilde{\mathbb{G}}^*(\mathbb{A}_{\mathbb{F}}))$ satisfying the following conditions:

- $\phi_{v_0} = f_0$;
- For any $1 \leq i \leq m$, at one place w above v_i , $\phi_w = f_{\text{St}}$ is a matrix coefficient of the Steinberg representation and at the remaining $r-1$ places above v_i , the corresponding local factor of ϕ equals to an idempotent e_{St} in the local Hecke algebra such that $f_{\text{St}} * e_{\text{St}} = f_{\text{St}}$;
- At one place w above v_{m+1} , $\phi_w = f_\rho$ is a matrix coefficient of the supercuspidal representation ρ and at the remaining $r-1$ places above v_{m+1} , the local factor of ϕ equals to the idempotent e_ρ ;
- At the unique place w_{m+2} above v_{m+2} , $\phi_{w_{m+2}}$ is a pseudo-coefficient of the Steinberg representation.
- Let S be a finite set of finite primes of $\mathbb{F}$ disjoint from all the places fixed above, and containing an additional prime v_{m+3} of $\mathbb{F}$ distinct from $\{v_i \mid 0 \leq i \leq m+2\}$ and totally split $\tilde{\mathbb{F}}$. For every $v \in S$ we consider a pair of matching functions ϕ_v, f_v which are supported on the set of (σ) -regular elements, and assume that $\phi_{v_{m+3}}, f_{v_{m+3}}$ are supported on the (σ) -elliptic elements.

By [AC89, Lemma 2.5], we have the simple twisted trace formula

$$(2.11) \quad \sum_{\tilde{\Pi}^*} \text{Tr}(\phi I_\sigma | \tilde{\Pi}^*) = \sum_{\delta} c_\delta TO_{\delta\sigma}(\phi)$$

where the left hand side runs over σ -stable cuspidal automorphic representations of $\tilde{\mathbb{G}}^*$, the right hand side runs over semisimple conjugacy classes δ in $\tilde{\mathbb{G}}^*(\mathbb{F}) = \text{GL}_n(\tilde{\mathbb{F}})$ such that $\mathcal{N}_r\delta$ is regular elliptic in $\mathbb{G}^*(\mathbb{F}) = \text{GL}_n(\mathbb{F})$ and c_δ is a certain nonzero volume factor.

Note that any $\tilde{\Pi}^*$ which contributes to the left hand side belongs to $\mathcal{A}^*$. By our choice of ϕ , the left hand side vanishes. We can choose $\delta \in \mathbb{G}^*(\tilde{\mathbb{F}})$ satisfying:

- At v_0 , δ is sufficiently close to δ_0 in $\tilde{\mathbb{G}}(\mathbb{F}_{v_0})$ so that $TO_{\delta\sigma}(\phi_{v_0}) \neq 0$;
- For $1 \leq i \leq m$ or $i = m+2$, at any place w above v_i , δ is strongly σ -regular and σ -elliptic, and is sufficiently close to 1 above v_{m+1} so that $TO_{\delta\sigma}(\phi_{v_i}) \neq 0$ (by our choice of ϕ_{v_i} , $TO_{\delta\sigma}(\phi_{v_i})$ equals to the character of the Steinberg or the supercuspidal representation ρ at $\mathcal{N}_r\delta$).

For functions having given support, the number of terms appearing on the right hand side of (2.11) is finite. By enlarging the set S if necessary and by choosing the functions ϕ_v, f_v for $v \in S$ appropriately, we can arrange that the only term which appears is our chosen δ (see [Clo90, p. 295] and [KR00, p. 812-813]). Then the right hand side of the simple twisted trace formula has precisely one nonzero term corresponding to δ and we reach a contradiction. $\square$

2.4.8. Finishing the proof of Theorem 2.4.1. Let $(L^*, D^*) \in \mathfrak{S}(G_r^*)$ be a σ -stable pair corresponding to $(L, D) \in \mathfrak{S}(G_r)$ under the Jacquet-Langlands transfer. Let π_L be an irreducible discrete series representation in the orbit D . We know from Proposition 2.4.8 and Lemma 2.4.6 that the set of σ -stable irreducible generic representations in the component $V(L, D)$ of the L^2 -Bernstein variety for G_r is Zariski dense. This implies that all irreducible generic representations in $V(L, D)$ are σ -stable. Indeed, for any $\phi \in C_c^\infty(G_r)$, the trace of ϕ and ϕ^σ define regular function on the variety $V(L, D)$, where $\phi^\sigma(g) := \phi(\sigma^{-1}(g))$. Since they coincide on a Zariski dense subset, they are equal on $V(L, D)$. This means that the character of any irreducible generic representation in $V(L, D)$ is σ -stable and hence the representation is σ -stable by Remark 2.4.1.

Consequently the G_r -conjugacy class of (L, D) is σ -stable. Let P be a parabolic subgroup of G_r containing L . Then there exists $w \in G_r$ such that $w\sigma^{-1}(P)w^{-1} = P$, $w\sigma^{-1}(L)w^{-1} = L$, together with an operator I_L^σ on π_L such that

$$I_L^\sigma \pi_L(w\sigma^{-1}(h)w^{-1}) = \pi_L(h)I_L^\sigma, \quad \forall h \in P.$$

Define an intertwining operator I_σ on the induced representation $\pi = i_P^G \pi_L$ by

$$(I_\sigma f)(g) := I_L^\sigma f(w\sigma^{-1}(g)), \quad \forall g \in G_r.$$

Then we have $I_\sigma \pi(g) = \pi(\sigma(g)) I_\sigma$ for all $g \in G_r$. Moreover, the same formula defines an intertwining operator, still denoted I_σ , on (the Jacquet-Langlands transfers of) any irreducible generic representation in the component $V(L, D)$, since such representations are full induced modules by Zelevinsky's theorem ([Zel80, 9.7]).

For any $\phi \in C_c^\infty(G_r)$, the trace of ϕI_σ defines a regular function on the variety $V(L, D)$. Therefore, the twisted character identity for any representation in $V(L, D)$ can be reduced to representations in the Zariski dense subset $\mathcal{A}_{v_0} \cap V(L, D)$ that we proved in Lemma 2.4.6. Therefore, we get

$$\mathrm{Tr}(\phi I_\sigma | \Pi) = e(\mathbf{G}_r) \mathrm{Tr}(\phi^* | \pi^*)$$

for any irreducible tempered representation π^* of G^* whose base change Π^* has a Langlands-Jacquet transfer Π (in the sense of Badulescu), which is an irreducible admissible representation of G_r .

2.5. Transfer of the Bernstein center. We first recall some standard facts about the (stable) Bernstein center from [Hai14]. In what follows, we work over a p -adic field F , freely using the analogues of the statements above which were proved for the case $F = \mathbb{Q}_p$. In this context F_r denotes the unique degree r unramified extension of F in a fixed algebraic closure of F . To simplify notation, we will use regular roman letters to denote both algebraic groups and their F -points (bearing in mind G_r is also the set of F -points for a Weil-restricted group over F).

Let ${}^L G = \hat{G} \rtimes W_F$ be the Langlands dual group of G . A *semisimple L -parameter* for G is a continuous homomorphism $\lambda : W_F \rightarrow {}^L G$ whose image consists of semisimple elements and whose projection to the second factor is the identity map. The set of $\hat{G}$ -conjugacy classes of semisimple L -parameters has the structure of an infinite disjoint union of algebraic varieties, which we refer to as the *stable Bernstein variety* of G and denote by $\mathfrak{Y}_G^{\mathrm{st}}$. The *stable Bernstein center* $\mathfrak{Z}_G^{\mathrm{st}}$ is defined to be the ring of regular functions on $\mathfrak{Y}_G^{\mathrm{st}}$. In particular, the stable Bernstein center for G is canonically identified with the stable Bernstein center for its quasi-split inner form G^* . For any $Z \in \mathfrak{Z}_G^{\mathrm{st}}$, we sometimes denote by Z^* the corresponding element in $\mathfrak{Z}_{G^*}^{\mathrm{st}}$.

There is a morphism of algebraic varieties $\mathfrak{Y}_G^{\mathrm{st}} \rightarrow \mathfrak{Y}_{G_r}^{\mathrm{st}}$ defined by restricting a semisimple L -parameter to the subgroup $W_{F_r} \subset W_F$. This induces the base change homomorphism of stable Bernstein centers:

$$(2.12) \quad b : \mathfrak{Z}_{G_r}^{\mathrm{st}} \rightarrow \mathfrak{Z}_G^{\mathrm{st}}$$

In our setting, since G^* is a product of Weil restrictions of general linear groups, the local Langlands correspondence is known and there is a quasi-finite map from the stable Bernstein center $\mathfrak{Z}_G^{\mathrm{st}}$ to the Bernstein center $\mathfrak{Z}_G$ which is an isomorphism if $G = G^*$. A good reference for these facts is [Coh18]. In particular, we can view an element $Z \in \mathfrak{Z}_G^{\mathrm{st}}$ as a distribution on the group G acting on any irreducible smooth representation π of G by a scalar $Z(\pi)$. In particular, if π^* is the Jacquet-Langlands transfer of π , then $Z^*(\pi^*) = Z(\pi)$. Moreover, for any $Z \in \mathfrak{Z}_{G_r}^{\mathrm{st}}$ and any tempered irreducible representation π^* of G^* we have

$$Z(BC(\pi^*)) = b(Z)(\pi^*)$$

where $BC(\pi^*)$ is the base change of π^* whose existence is established in [AC89].

Theorem 2.5.1. *Let $\phi \in C_c^\infty(G_r)$ and let $\phi^* \in C_c^\infty(G^*)$ be its stable base change transfer. Then for any $Z \in \mathfrak{Z}_{G_r}^{\mathrm{st}}$, $b(Z)^* * \phi^*$ is a stable base change transfer of $Z * \phi$.*

Proof. Let ϕ^* be a stable base change transfer of ϕ and let $(Z * \phi)^*$ be a stable base change transfer of $Z * \phi$. By Kazhdan's density theorem it suffices to show that

$$\mathrm{Tr}(b(Z)^* * \phi^* | \pi^*) = \mathrm{Tr}((Z * \phi)^* | \pi^*)$$

for any irreducible tempered representation π^* of G^* .

Fix such a π^* . Let Π^* be the base change lifting of π^* to G_r^* as in [AC89, Theorem 6.2]. If Π^* does not have a Langlands-Jacquet transfer to G_r , then both sides are 0 by Theorem 2.4.1(2).

Now suppose that Π^* has a Langlands-Jacquet transfer Π , which is an irreducible admissible representation of G_r . By part (1) of Theorem 2.4.1 we know that Π is σ -stable and we can choose the intertwining operator I_σ so that

$$\begin{aligned} \mathrm{Tr}((Z * \phi)^* | \pi^*) &= e(G_r) \mathrm{Tr}((Z * \phi) I_\sigma | \Pi) = e(G_r) Z(\Pi) \mathrm{Tr}(\phi I_\sigma | \Pi) \\ &= e(G_r) Z(\Pi^*) \mathrm{Tr}(\phi I_\sigma | \Pi) = b(Z)^*(\pi^*) \mathrm{Tr}(\phi^* | \pi^*) = \mathrm{Tr}(b(Z)^* * \phi^* | \pi^*). \end{aligned}$$

□

Lemma 2.5.2. *Let $\phi \in C_c^\infty(G(F_r))$ and let $\phi^* \in C_c^\infty(G^*(F))$ be its stable base change transfer. Then ϕ has the vanishing property if and only if $\mathrm{Tr}(\phi^* | \pi^*) = 0$ for any irreducible tempered representation π^* of $G^*(F)$ that does not have a Langlands-Jacquet transfer to $G(F)$.*

Proof. We start with some preliminary remarks. Any irreducible tempered representation π of $G(F)$ (resp. π^* of $G^*(F)$) can be written as a normalized induction $i_{P(F)}^{G(F)}(\sigma)$ (resp. $i_{P^*(F)}^{G^*(F)}(\sigma^*)$) of an essentially discrete series representation σ (resp. σ^*) of $M(F)$ (resp. $M^*(F)$) for a Levi factor $M \subset P$ (resp. $M^* \subset P^*$); see [Tad90]. The property that π^* does not have a Langlands-Jacquet transfer to $G(F)$ is equivalent to the condition that M^* does not transfer to G (comp. [Bad03, p. 105]). Indeed, if M^* does transfer to an F -Levi $M \subset G$, then σ^* has a discrete series transfer σ on $M(F)$, in which case π^* is the Jacquet-Langlands transfer of the tempered representation $i_{P(F)}^{G(F)}(\sigma)$ for any F -parabolic P having M as F -Levi factor (see [Bad18, Prop. 7.1(b)]). Conversely, if M^* does not transfer, then π^* cannot be the Jacquet-Langlands transfer of a tempered irreducible representation $\pi = i_{Q(F)}^{G(F)}(\sigma)$, where $Q = LU$ is a Levi factorization of an F -parabolic and σ is a discrete series on $L(F)$. If π^* were the transfer of such a π , then we would have $\pi^* = i_{Q^*(F)}^{G^*(F)}(\sigma^*)$ where σ^* is a discrete series transfer of σ , and where $Q^* = L^*U^*$ is a transfer of $Q = LU$. By uniqueness of the discrete series inducing data of tempered representations of G^* ([Zel80, Thm. 9.7]), we must have $M = L^*$, contradicting our assumption that M does not transfer.

In a slightly different direction, Badulescu defined an injective map on representation rings

$$\mathrm{JL} : R(G) \rightarrow R(G^*),$$

and termed the condition that π^* not lie in the image by $\pi^* \in \mathbf{S}_{G^*, G}$. Badulescu also proved that, for $\phi^* \in \mathcal{H}(G^*(F))$, the property that

$$\mathrm{Tr}(\phi^* | \pi^*) = 0, \quad \forall \pi^* \in \mathbf{S}_{G^*, G}$$

is equivalent to the property that

$$O_{\gamma^*}(\phi^*) = 0, \quad \forall \gamma^* \in G^{*, \mathrm{rs}}(F) \setminus G(F)$$

that is, for all regular semisimple γ^* which do not transfer to $G(F)$; see [Bad03, Lem. 3.3]. This statement is clearly related to what we wish to prove, but it is not quite sufficient.

We now give the proof, which is quite analogous to [Coh18, Lem. 8.2]. First suppose that ϕ has the vanishing property. Then the orbital integral of ϕ^* vanishes on any regular semisimple conjugacy class γ^* of G^* that does not transfer to G . In fact, if such an element γ^* is not a norm from G_r then $O_{\gamma^*}(\phi^*) = 0$ by the definition of stable base change transfer; while if $\gamma^* = \mathcal{N}_r \delta$ for some $\delta \in G_r$ then $\mathcal{N}_r \delta$ is not G_r -conjugate to any element of G (since otherwise γ^* would transfer to G), so that $O_{\gamma^*}(\phi^*) = TO_{\delta \sigma}(\phi) = 0$ by the vanishing property of ϕ . By the Weyl integration formula, this implies that $\mathrm{Tr}(\phi^* | \pi^*) = 0$ for any tempered irreducible representation π^* of G^* that does not have a Langlands-Jacquet transfer to G . In fact, by the preliminary discussion above, any such representation π^* is a representation induced from a Levi subgroup $M^* \subset G^*$ that does not transfer to G , and the character of such a representation has (its regular-semisimple) support consisting of regular semisimple elements γ^* that are conjugate to elements in M^* and hence do not transfer to $G(F)$. Here we use the observation in [Bad03, p. 105] that if M^* does not transfer to $G(F)$, then no element of $G^{*, \mathrm{rs}}(F) \cap M^*$ can transfer. (Alternatively, one can use the argument at the end of the proof of Lemma 2.4.2.)

Next we prove the implication in the reverse direction. Assume that $\text{Tr}(\phi^*|\pi^*) = 0$ for any representation π^* of G^* that does not have a Langlands-Jacquet transfer to G . Suppose $\delta \in G(F_r)$ is a σ -semisimple element such that $N_r\delta \in G(F_r)$ is not conjugate to any element in $G(F)$. There is a semisimple element $\gamma^* \in G^*(F)$ that has the same characteristic polynomial as $N_r\delta$. Then the conjugacy class of γ^* does not transfer to $G(F)$. Note that γ^* is an elliptic element in an F -Levi subgroup M^* of G^* , and M^* does not transfer to an F -Levi subgroup in G (because elliptic tori transfer to any inner form of M^* , see for example [Kot86b, §10]).

Let P^* be an F -parabolic subgroup of G^* with F -Levi factor M^* . Any irreducible tempered representation π^* of M^* is induced from a discrete series representation of a Levi subgroup L^* of M^* . By induction in stages we see that $i_{P^*(F)}^{G^*(F)}\pi^*$ is tempered, and it is irreducible by a result of Bernstein-Zelevinsky. It does not have a Langlands-Jacquet transfer to $G(F)$: this is because M^* does not transfer to $G(F)$; use [Bad18, Prop. 7.1(a)]. Hence

$$\text{Tr}((\phi^*)^{P^*}|\pi^*) = \text{Tr}(\phi^*|i_{P^*(F)}^{G^*(F)}\pi^*) = 0$$

by assumption and the character descent formula. This implies that $O_{\gamma^*}^{M^*}((\phi^*)^{P^*}) = 0$ by Kazhdan's density theorem and hence $O_{\gamma^*}(\phi^*) = 0$ by the descent formula for orbital integrals. Then we get

$$TO_{\delta\sigma}(\phi) = \pm O_{\gamma^*}(\phi^*) = 0$$

from which we conclude that ϕ has the vanishing property. $\square$

Definition 2.5.3. Let $\phi \in C_c^\infty(G_r)$. A function $f \in C_c^\infty(G)$ is called a *base change transfer* of ϕ if for any semisimple element $\gamma \in G$, we have $e(G_\gamma)O_\gamma(f) = e(G_{\delta\sigma})TO_{\delta\sigma}(\phi)$ if γ is G_r -conjugate to the naive norm $N_r\delta$ of some σ -semisimple element $\delta \in G_r$, and $O_\gamma(f) = 0$ otherwise.

Lemma 2.5.4. *If $\phi \in C_c^\infty(G_r)$ has the vanishing property, then it has a base change transfer $f \in C_c^\infty(G)$.*

Proof. Let $\phi^* \in C_c^\infty(G^*)$ be a stable base change transfer of ϕ . Let $\gamma^* \in G^*$ be a semisimple element that is not stably conjugate to any element of G . If γ^* is not the norm of any element in G_r , then by definition $O_{\gamma^*}(\phi^*) = 0$. If $\gamma^* = N_r\delta$ for some $\delta \in G_r$, then by the vanishing property we have $O_{\gamma^*}(\phi^*) = \pm TO_{\delta\sigma}(\phi) = 0$. Thus ϕ^* has a Jacquet-Langlands transfer $f \in C_c^\infty(G)$ by [DKV84]. Then f is a base change transfer of ϕ by definition. $\square$

Proposition 2.5.5. *If $\phi \in C_c^\infty(G_r)$ has the vanishing property (in the sense of Definition 2.1.6), then so does $Z * \phi$ for any $Z \in \mathfrak{Z}_{G_r}^{\text{st}}$.*

Proof. Let ϕ^* be the stable base change transfer of ϕ and let $(Z * \phi)^* \in C_c^\infty(G(F))$ be the stable base change transfer of $Z * \phi$. By Lemma 2.5.2, it suffices to show that

$$\text{Tr}((Z * \phi)^*|\pi^*) = 0$$

for any irreducible tempered representation π^* of G^* that does not have a Jacquet-Langlands transfer to G .

Fix such a π^* . Let Π^* be the base change of π^* to G_r^* as in [AC89, Theorem 6.2]. If Π^* does not have a Langlands-Jacquet transfer to G_r , then we are done by Theorem 2.4.1.

Now suppose that Π^* has a Langlands-Jacquet transfer Π , which is an irreducible tempered representation of G_r . By part (1) of Theorem 2.4.1 we know that Π is σ -stable and we choose the intertwining operator I_σ as in *loc. cit* so that

$$\text{Tr}((Z * \phi)^*|\pi^*) = \text{Tr}((Z * \phi)I_\sigma|\Pi).$$

This vanishes since the right hand side is a scalar multiple of

$$\text{Tr}(\phi I_\sigma|\Pi) = e(G_r)\text{Tr}(\phi^*|\pi^*) = 0.$$

Here the first equality follows from Theorem 2.4.1 and the second equality follows from Lemma 2.5.2. $\square$

We mention the following Corollary.

Corollary 2.5.6. *Let $\mathcal{G}$ be a smooth group scheme over $\mathcal{O}$ with geometrically connected fibers such that the generic fiber of $\mathcal{G}$ is isomorphic to G . Then $Z * 1_{\mathcal{G}(\mathcal{O}_r)}$ has the vanishing property for any $Z \in \mathfrak{Z}_{G_r}^{\text{st}}$.*

Proof. This follows from Proposition 2.5.5 and a result in [Kot86a]. More precisely the group $\mathcal{G}(\check{F})$, where $\check{F}$ is the completion of the maximal unramified extension of F , satisfies the properties (a),(b), (c) on page 240 of *loc. cit.* Hence by the discussion after the Corollary on page 244 of *loc. cit.* we get that $1_{\mathcal{G}(\mathcal{O}_r)}$ has the vanishing property. $\square$

Theorem 2.5.7. *If $\phi \in C_c^\infty(G_r)$ satisfies the vanishing property and has a base change transfer $f \in C_c^\infty(G)$, then $b(Z) * f$ is a base change transfer of $Z * \phi$ for any $Z \in \mathfrak{Z}_{G_r}^{\text{st}}$.*

Proof. Let $\phi^* \in C_c^\infty(G^*)$ be the stable base change of ϕ . Then it is a Jacquet-Langlands transfer of f , as follows from the definitions. Thus $b(Z)^* * \phi^*$ is a Jacquet-Langlands transfer of $b(Z) * f$, by the case $r = 1$ of Theorem 2.5.1. On the other hand, $b(Z)^* * \phi^*$ is a stable base change transfer of $Z * \phi$ by Theorem 2.5.1. Keeping in mind that $Z * \phi$ satisfies the vanishing property by Proposition 2.5.5, this implies that $b(Z) * f$ is a base change transfer of $Z * \phi$. $\square$

3. p -DIVISIBLE GROUPS WITH EL STRUCTURE AND TEST FUNCTIONS

In this section we follow [Sch13b, §3, §4] and generalize some results there to our situation.

3.1. EL data. We start with a list of data in the local setting. Let B be a semisimple $\mathbb{Q}_p$ -algebra and V a finitely generated left B -module. We get a semisimple $\mathbb{Q}_p$ -algebra $C := \text{End}_B(V)$ and an algebraic group G over $\mathbb{Q}_p$ whose points in any $\mathbb{Q}_p$ -algebra R is defined by $G(R) = (C \otimes_{\mathbb{Q}_p} R)^\times$, the group of units of the algebra $C \otimes_{\mathbb{Q}_p} R$.

Let $\bar{\mu}$ be a $G(\bar{\mathbb{Q}}_p)$ -conjugacy class of cocharacters $\mu : \mathbb{G}_m \rightarrow G_{\bar{\mathbb{Q}}_p}$. The local reflex field $E \subset \bar{\mathbb{Q}}_p$ is defined as the field of definition of the conjugacy class $\bar{\mu}$. Let $\mathcal{O}_E \subset E$ be the ring of integers and κ_E the residue field. We assume that $\bar{\mu}$ is *minuscule*. In other words, after choosing a representative μ of $\bar{\mu}$, only weight 0 or 1 occurs in the corresponding weight decomposition of $V_{\bar{\mathbb{Q}}_p}$. Fix such a representative μ and write $V_{\bar{\mathbb{Q}}_p} = V_1 \oplus V_2$ where $\mu(t)$ acts by t (resp. 1) on V_1 (resp. V_2) for each $t \in \bar{\mathbb{Q}}_p^\times$.

We also fix integral data consisting of a maximal $\mathbb{Z}_p$ -order $\mathcal{O}_B \subset B$ and a periodic chain $\mathcal{L} = \{\Lambda_i, i \in \mathbb{Z}\}$ of $\mathcal{O}_B$ -stable $\mathbb{Z}_p$ -lattices in V . These allow us to define a $\mathbb{Z}_p$ -model $\mathcal{G}_{\mathcal{L}}$ of G whose points in any $\mathbb{Z}_p$ -algebra R is $\mathcal{G}_{\mathcal{L}}(R) := \text{Aut}_{\mathcal{O}_B} \mathcal{L}_R$. Here $\mathcal{L}_R := \{\Lambda \otimes_{\mathbb{Z}_p} R, \Lambda \in \mathcal{L}\}$ and $\text{Aut}_{\mathcal{O}_B} \mathcal{L}_R$ consists of a family of automorphisms of the $\mathcal{O}_B \otimes_{\mathbb{Z}_p} R$ -modules $\Lambda \otimes_{\mathbb{Z}_p} R$ for each $\Lambda \in \mathcal{L}$ that preserves the natural morphisms among the lattices in $\mathcal{L}$. In particular, $\mathcal{G}$ is a product of Weil restrictions of inner forms of general linear groups and $\mathbf{P}_{\mathcal{L}} := \mathcal{G}_{\mathcal{L}}(\mathbb{Z}_p)$ is a parahoric subgroup of $G(\mathbb{Q}_p)$. More generally for each integer $r \geq 1$, we denote $\mathbf{P}_{\mathcal{L}, r} := \mathcal{G}_{\mathcal{L}}(\mathbb{Z}_{p^r})$ the corresponding parahoric subgroup of $G(\mathbb{Q}_{p^r})$. We let $\mathcal{D} = (B, V, \mathcal{O}_B, \mathcal{L}, \bar{\mu})$ denote this list of data.

3.2. Deformation spaces of p -divisible groups.

Definition 3.2.1. For any scheme S on which p is locally nilpotent, a p -divisible $\mathcal{O}_B$ -module over S is a pair (H, i) consisting of a p -divisible group H over S together with a $\mathbb{Z}_p$ -algebra homomorphism $i : \mathcal{O}_B \rightarrow \text{End}(H)$. An isogeny or quasi-isogeny between p -divisible $\mathcal{O}_B$ -modules is required to commute with the $\mathcal{O}_B$ -action.

Definition 3.2.2. Let S be an $\mathcal{O}_E$ -scheme on which p is locally nilpotent. A p -divisible group with $\mathcal{D}$ -structure $H_\bullet$ over S consists of the data of

- A collection of p -divisible $\mathcal{O}_B$ -modules H_Λ for each $\Lambda \in \mathcal{L}$. Denote the Lie algebra of the universal vector extension of H_Λ by M_Λ ;
- For each inclusion $\Lambda \subset \Lambda'$ of lattices in $\mathcal{L}$, an isogeny $\rho_{\Lambda', \Lambda} : H_\Lambda \rightarrow H_{\Lambda'}$

satisfying the following conditions

- (1) For any triple of lattices $\Lambda \subset \Lambda' \subset \Lambda''$ we have $\rho_{\Lambda'', \Lambda'} \circ \rho_{\Lambda', \Lambda} = \rho_{\Lambda'', \Lambda}$;

- (2) For any inclusion of lattices $\Lambda \subset \Lambda'$, the cokernel of the induced map $M_\Lambda \rightarrow M_{\Lambda'}$ is locally on S isomorphic to $(\Lambda'/\Lambda) \otimes_{\mathbb{Z}_p} \mathcal{O}_S$ as $\mathcal{O}_B \otimes_{\mathbb{Z}_p} \mathcal{O}_S$ -modules;
- (3) For each Λ , we have an equality of polynomial functions on $\mathcal{O}_B$:

$$\det_{\mathcal{O}_S}(b; \text{Lie}(H_\Lambda)) = \det(b; V_1).$$

- (4) For any $b \in B^\times$ that normalizes $\mathcal{O}_B$ and any $\Lambda \in \mathcal{L}$, a periodicity isomorphism $\theta_b : H_\Lambda^b \rightarrow H_{b\Lambda}$. Here H_Λ^b is the p -divisible $\mathcal{O}_B$ -module with the same underlying p -divisible group as H_Λ but with the action map $i_\lambda : \mathcal{O}_B \rightarrow \text{End}(H_\Lambda)$ replaced by the composition of i_λ with the automorphism of $\mathcal{O}_B$ defined by $x \mapsto bxb^{-1}$.

Remark 3.2.3. By the discussion in [RZ96, §3.23 c)], condition (3) implies that for each $\Lambda \in \mathcal{L}$, M_Λ is locally on S isomorphic to $\Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}_S$ as an $\mathcal{O}_B \otimes_{\mathbb{Z}_p} \mathcal{O}_S$ -module.

Let κ be a perfect field of characteristic p , viewed as an $\mathcal{O}_E$ -algebra via a homomorphism $\mathcal{O}_E \rightarrow \kappa$. Let $H_\bullet$ be a p -divisible group with $\mathcal{D}$ -structure over κ .

Definition 3.2.4. The deformation space $\mathcal{X}_{H_\bullet}$ of $H_\bullet$ is the functor that associates to any Artin local $\mathcal{O}_E$ -algebra R with residue field κ the set of isomorphism classes of p -divisible groups with $\mathcal{D}$ -structure $\tilde{H}_\bullet$ over $\text{Spec}(R)$, together with an isomorphism $\tilde{H}_\bullet \otimes_R \kappa \xrightarrow{\cong} H_\bullet$, i.e. an isomorphism of p -divisible $\mathcal{O}_B$ -modules $\tilde{H}_\Lambda \otimes_R \kappa \xrightarrow{\cong} H_\Lambda$ for all $\Lambda \in \mathcal{L}$ that commute with the isogenies corresponding to the inclusions among various Λ 's.

Theorem 3.2.5. *The functor $\mathcal{X}_{H_\bullet}$ is representable by a complete noetherian local $\mathcal{O}_E$ -algebra $R_{H_\bullet}$ with residue field κ .*

Proof. By [Sch13b, Theorem 3.6], the deformation functor of each single H_Λ is representable by a complete noetherian local $\mathcal{O}_E$ -algebra R_{H_Λ} with residue field κ . Then by the periodicity condition (4), the functor $\mathcal{X}_{H_\bullet}$ is representable by a quotient of the completed tensor product of finitely many such $\mathcal{O}_E$ -algebras. $\square$

Let $X_{H_\bullet}$ be the rigid generic fiber of $\mathcal{X}_{H_\bullet}$. Then $X_{H_\bullet}$ is a rigid analytic space over the discrete valued field $W_{\mathcal{O}_E}(\kappa)[\frac{1}{p}]$, where $W_{\mathcal{O}_E}(\kappa) := W(\kappa) \otimes_{\mathbb{Z}_p} \mathcal{O}_E$. Moreover, it is smooth by the same reasoning in [Sch13b, Theorem 3.8]. For each compact open subgroup $K \subset \mathcal{G}_{\mathcal{L}}(\mathbb{Z}_p)$, we have an étale cover $X_{H_\bullet, K}$ of $X_{H_\bullet}$ parametrizing level- K structures of the universal p -adic Tate module.

For each lattice $\Lambda \in \mathcal{L}$, the covariant Dieudonné module M_Λ is an $\mathcal{O}_B \otimes_{\mathbb{Z}_p} W(\kappa)$ -module that is free of finite rank as $W(\kappa)$ -module. Moreover, the $B \otimes_{\mathbb{Q}_p} W(\kappa)[\frac{1}{p}]$ -modules $M_\Lambda[\frac{1}{p}]$ for various Λ 's are canonically identified by the isogenies among them and we denote this common object by $V_{H_\bullet}$. Now suppose $X_{H_\bullet} \neq \emptyset$; then $V_{H_\bullet}$ is isomorphic to $V \otimes_{\mathbb{Z}_p} W(\kappa)$ as B -module. We can choose an isomorphism $V_{H_\bullet} \cong V \otimes_{\mathbb{Z}_p} W(\kappa)$ such that the $\mathcal{O}_B$ -module M_Λ is mapped isomorphically to $\Lambda \otimes_{\mathbb{Z}_p} W(\kappa)$ for each $\Lambda \in \mathcal{L}$. Under this identification, the Frobenius operator F on $V_{H_\bullet}$ has the form $F = p\delta\sigma$ where σ denotes the Frobenius automorphism of $W(\kappa)$ and $\delta \in G(W(\kappa)[\frac{1}{p}])$ satisfies $p\Lambda \subset p\delta\Lambda \subset \Lambda$ for all $\Lambda \in \mathcal{L}$. A different choice of isomorphism changes δ by a σ -conjugate under $\mathcal{G}_{\mathcal{L}}(W(\kappa))$.

To state a further condition satisfied by δ we introduce more notations. Let $B(G)$ be the set of σ -conjugacy classes in $G(W(\bar{\kappa})[\frac{1}{p}])$. In [Kot90, §6] Kottwitz constructs a map $\kappa_G : B(G) \rightarrow X^*(Z(\hat{G})^\Gamma)$ where $\Gamma = \text{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$. We choose a representative μ of the conjugacy class $\bar{\mu}$ and view it as a character of the dual group. Then the restriction $\mu_1 \in X^*(Z(\hat{G})^\Gamma)$ of μ depends only on the conjugacy class $\bar{\mu}$.

Proposition 3.2.6. *For any perfect field κ of characteristic p equipped with a ring homomorphism $\mathcal{O}_E \rightarrow \kappa$, the association $H_\bullet \mapsto \delta \in G(W(\kappa)[\frac{1}{p}])$ described above defines an injection from the set of p -divisible groups with $\mathcal{D}$ -structures $H_\bullet$ over κ such that $X_{H_\bullet} \neq \emptyset$ to the set of $\mathcal{G}_{\mathcal{L}}(W(\kappa))$ - σ -conjugacy classes $\delta \in G(W(\kappa)[\frac{1}{p}])$ such that $\kappa_G(p\delta) = -\mu_1$ and $p\Lambda_{W(\kappa)} \subset p\delta\Lambda_{W(\kappa)} \subset \Lambda_{W(\kappa)}$ for all $\Lambda \in \mathcal{L}$, where $\Lambda_{W(\kappa)} := \Lambda \otimes_{\mathbb{Z}_p} W(\kappa)$.*

Proof. The injectivity follows from classical Dieudonné theory. The identity $\kappa_G(p\delta) = -\mu_1$ follows from [RZ96, Proposition 1.20], see also [Sch13b, Proposition 3.9]. $\square$

3.3. Test functions. Fix an integer $j \geq 1$ and an element $\tau \in \text{Frob}^j I_E \subset W_E$ in the Weil group of E (here I_E is the inertia subgroup of W_E and $\text{Frob} \in W_E$ is a geometric Frobenius element). Set $r = j[\kappa_E : \mathbb{F}_p]$ and view $\mathbb{F}_{p^r}$ as a degree j extension of κ_E . Let $h \in C_c^\infty(\mathcal{G}_{\mathcal{L}}(\mathbb{Z}_p))$ be a $\mathbb{Q}$ -valued function. Our goal is to define a test function $\phi_{\tau,h}$ in $C_c^\infty(G(\mathbb{Q}_{p^r}))$ that enters the Lefschetz trace formula for the special fiber at p of the relevant Shimura varieties, and also a test function $\phi_h^{(r)}$ that is relevant for the description of semi-simple local factor of the Shimura varieties.

Let $H_\bullet$ be a p -divisible group with $\mathcal{D}$ -structure over a perfect field κ of characteristic p equipped with a ring homomorphism $\mathcal{O}_E \rightarrow \kappa$. Recall that for any compact open subgroup $K \subset \mathcal{G}_{\mathcal{L}}(\mathbb{Z}_p)$, we defined a rigid analytic space $X_{H_\bullet, K}$ over the field $k := W_{\mathcal{O}_E}(\kappa)[\frac{1}{p}]$. Following [Sch13b, Definition 3.11], we say that $H_\bullet$ has *controlled cohomology* if $X_{H_\bullet, K}$ has controlled ℓ -adic étale cohomology for any normal pro- p open subgroup $K \subset \mathcal{G}_{\mathcal{L}}(\mathbb{Z}_p)$ and any prime $\ell \neq p$. We refer to [Sch13b, §2] for the notion of rigid analytic spaces with controlled cohomology.

Definition 3.3.1. Let $\delta \in G(\mathbb{Q}_{p^r})$. If δ is associated to some p -divisible group with $\mathcal{D}$ -structure $H_\bullet$ over $\mathbb{F}_{p^r}$ as in Proposition 3.2.6 such that $H_\bullet$ has controlled cohomology, we define

$$\phi_{\tau,h}(\delta) = \text{tr}(\tau \times h | H^*(X_{H_\bullet, K} \otimes_k \hat{k}, \mathbb{Q}_\ell))$$

and

$$\phi_h^{(r)}(\delta) = \text{tr}^{\text{ss}}(\text{Frob}^j \times h | H^*(X_{H_\bullet, K} \otimes_k \hat{k}, \mathbb{Q}_\ell))$$

where $K \subset \mathcal{G}_{\mathcal{L}}(\mathbb{Z}_p)$ is any normal pro- p open compact subgroup such that h is K -biinvariant and $\ell \neq p$. For any other $\delta \in G(\mathbb{Q}_{p^r})$, define $\phi_{\tau,h}(\delta)$ and $\phi_h^{(r)}(\delta)$ to be 0.

Here tr^{ss} denotes semi-simple trace, for whose definition and properties we refer to [HN02, §3]. In particular, it follows by definition that for all $\delta \in G(\mathbb{Q}_{p^r})$ we have

$$(3.1) \quad \phi_h^{(r)}(\delta) = \int_{\text{Frob}^j I_E} \phi_{\tau,h}(\delta) d\tau$$

where $d\tau$ is the translate of the Haar measure on I_E having total volume 1; comp., [HR20, §8.2]. Since the alternating sum $H^*(X_{H_\bullet, K} \otimes_k \hat{k}, \mathbb{Q}_\ell)$ is a virtual continuous ℓ -adic representation of the Weil group W_E , the action of the inertia I_E on its semi-simplification factors through a finite quotient and the above integral reduces to a finite sum multiplied by a nonzero rational number.

Proposition 3.3.2. *The functions $\phi_{\tau,h}$ and $\phi_h^{(r)}$ are well-defined locally constant $\mathbb{Q}$ -valued functions with compact support on $G(\mathbb{Q}_{p^r})$ and they are independent of ℓ . Moreover, $\phi_{\tau,h}$ and $\phi_h^{(r)}$ are $\mathcal{G}_{\mathcal{L}}(\mathbb{Z}_{p^r})$ - σ -conjugation invariant and their supports are contained in the set of elements $\delta \in G(\mathbb{Q}_{p^r})$ such that $\kappa_G(p\delta) = -\mu_1$ and $p\Lambda_r \subset p\delta\Lambda_r \subset \Lambda_r$ for all $\Lambda \in \mathcal{L}$, where $\Lambda_r := \Lambda \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p^r}$.*

Proof. By (3.1), it suffices to prove the results for $\phi_{\tau,h}$. The same proof in [Sch13b, Proposition 4.2] shows that $\phi_{\tau,h}$ is well-defined, has value in $\mathbb{Q}$ and independent of ℓ . The local constancy is proved in the same way as [Sch13b, Proposition 4.3]. The second statement follows from Proposition 3.2.6. $\square$

Proposition 3.3.3. *The function $\phi_{\tau,h}$ depends only on the data $(\mathcal{G}_{\mathcal{L}}, \bar{\mu}, \tau, h)$, and is independent of the choice of EL -data $\mathcal{D}$ giving rise to them.*

Proof. We may assume that B is a simple algebra whose center F is a finite extension of $\mathbb{Q}_p$. Then we can choose an isomorphism $B \cong M_m(D)$ sending $\mathcal{O}_B$ to $M_m(\mathcal{O}_D)$ where D is a division algebra and $\mathcal{O}_D$ is the unique maximal order in D . Let $W = D^m$ be the unique simple left B -module. There is an isomorphism $V \cong W^n$ of B -modules which induces isomorphisms $C \cong M_n(D^{op})$ and $G \cong \text{Res}_{F/\mathbb{Q}_p} \text{GL}_n(D^{op})$. Let $\epsilon \in \mathcal{O}_B$ be the idempotent corresponding to the matrix whose $(1, 1)$ entry equals to 1 and all other entries equal to 0. Let $B' = D$ and $V' = \epsilon V \cong D^n$ viewed as a left B' -module. Then the group of B' -automorphisms of V' is canonically isomorphic to G . Let

$\mathcal{O}_{B'} = \mathcal{O}_D$ be the unique maximal order and $\mathcal{L}' = \epsilon\mathcal{L} := \{\epsilon\Lambda, \Lambda \in \mathcal{L}\}$ be the $\mathcal{O}_{B'}$ -lattice chain in V' corresponding to $\mathcal{L}$. Then the $\mathcal{O}$ -group scheme $\mathcal{G}_{\mathcal{L}'}$ constructed from the data $(\mathcal{O}_{B'}, \mathcal{L}')$ is canonically isomorphic to $\mathcal{G}_{\mathcal{L}}$. Let $\mathcal{D}' := (B', V', \mathcal{O}_{B'}, \mathcal{L}', \bar{\mu})$. Then there is an equivalence between the category of p -divisible groups with $\mathcal{D}$ -structure and those with $\mathcal{D}'$ -structures. Indeed, for any $H_{\bullet}$ in the former category, $H'_{\bullet} := \epsilon H_{\bullet}$ is an object in the latter category. Conversely, for any $H'_{\bullet}$ in the latter category, $H_{\bullet} := (H'_{\bullet})^m$ defines an object in the former category. In particular, corresponding objects defined over a perfect field κ of characteristic p are parametrized by the same invariant $\delta \in G(W(\kappa)[\frac{1}{p}])$, defined up to $\mathcal{G}_{\mathcal{L}}(W(\kappa))$ - σ -conjugacy and have isomorphic deformation spaces. Therefore the function $\phi'_{\tau,h}$ defined from the data $\mathcal{D}'$ is the same as the function $\phi_{\tau,h}$ and the claim follows. $\square$

3.4. Statement of the main local theorem. Choose a representative μ of the conjugacy class $\bar{\mu}$ of cocharacters of $G_{\bar{\mathbb{Q}}_p}$ and view it as a character of the standard maximal torus $\hat{T}$ in $\hat{G}$. Let $r_{-\mu} : \hat{G} \rightarrow \mathrm{GL}(V_{-\mu})$ be the irreducible algebraic representation of $\hat{G}$ with extreme $\hat{T}$ -weight $-\mu$. In other words, the extreme weight of $r_{-\mu}$ is the dominant representative in the Weyl group orbit of $-\mu$. By [Kot84, Lemma 2.1.2], for any choice of W_E -invariant splitting of $\hat{G}$, $r_{-\mu}$ extends uniquely to a representation

$$r_{-\mu} : {}^L G_E := \hat{G} \rtimes W_E \rightarrow \mathrm{GL}(V_{-\mu})$$

such that W_E acts as the identity on the corresponding highest weight space.

Fix $\tau \in \mathrm{Frob}_E^j I_E \subset W_E$ and set $r := j[\kappa_E : \mathbb{F}_p]$ as before.

Definition 3.4.1. Let $z_{\tau,-\mu}$ be the element in the stable Bernstein center of G whose value at any semisimple Langlands parameter $\varphi : W_{\mathbb{Q}_p} \rightarrow {}^L G = \hat{G} \rtimes \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ is given by

$$\mathrm{tr}(\tau|(r_{-\mu} \circ \varphi|_{W_E})| \cdot |_E^{-\langle \rho, \mu \rangle})$$

where ρ is half the sum of all positive roots of G . By [Hai14, §5.3]², this defines a regular function on the stable Bernstein variety.

Next we define the semisimple version of the element $z_{\tau,-\mu}$. Let $E_j \subset \bar{\mathbb{Q}}_p$ be the degree j unramified extension of E . Then we have $\tau \in W_{E_j} \subset W_E$. After identifying $\mathbb{Q}_{p^r}$ with the maximal unramified extension of $\mathbb{Q}_p$ in E_j , we get a natural inclusion $W_{E_j} \subset W_{\mathbb{Q}_{p^r}}$ which allows us to view τ also as an element in $W_{\mathbb{Q}_{p^r}}$. By [Hai14, §5.4] there is an element $z_{\tau,-\mu}^{(r)}$ in the stable Bernstein center of $G_{\mathbb{Q}_{p^r}}$ such that $b(z_{\tau,-\mu}^{(r)}) = z_{\tau,-\mu}$. In fact, for any irreducible smooth representation Π of $G(\mathbb{Q}_{p^r})$ with semisimple L -parameter

$$\varphi_{\Pi} : W_{\mathbb{Q}_{p^r}} \rightarrow \hat{G} \rtimes W_{\mathbb{Q}_{p^r}},$$

the element $z_{\tau,-\mu}^{(r)}$ acts on Π by the scalar

$$\mathrm{tr}(\tau|(r_{-\mu} \circ \varphi_{\Pi}|_{W_{E_j}})| \cdot |_E^{-\langle \rho, \mu \rangle}).$$

We also define a closely related element $z_{-\mu}^{(r)}$ in the stable Bernstein center of $G(\mathbb{Q}_{p^r})$ that acts on any irreducible smooth representation Π of $G(\mathbb{Q}_{p^r})$ with semisimple L -parameter φ_{Π} by the scalar

$$\mathrm{tr}(\mathrm{Frob}_E^j|(r_{-\mu} \circ \varphi_{\Pi}|_{W_{E_j}})^{I_E}| \cdot |_E^{-\langle \rho, \mu \rangle}).$$

Then we have

$$z_{-\mu}^{(r)} = \int_{\mathrm{Frob}_E^j I_E} z_{\tau,-\mu}^{(r)} d\tau$$

where $d\tau$ is the Haar measure of total volume 1 and the integral is actually a finite sum. When $r = 1$, we also simply denote $z_{-\mu} := z_{-\mu}^{(1)}$.

²Due to printing errors in the published version, all citations for this article refer to the online version, arXiv:1304.6293.

Remark 3.4.2. Above we are implicitly using the ring homomorphism from the stable Bernstein center to the usual Bernstein center; see [Hai14, §5.5]. In order to make sense of the Bernstein center (which requires a notion of normalized parabolic induction) and also in order to make sense of the ring homomorphism, we must fix a choice of $\sqrt{p} \in \bar{\mathbb{Q}}_\ell$. We use the same choice to define the symbol $|\cdot|_E^{(\rho, \mu)}$ appearing above. In this way we see that $z_{\tau, -\mu}$ and $z_{-\mu}^{(r)}$ are independent of the choice of $\sqrt{p}$. See [HR21, Thm. 7.15].

Now we can state our main local results.

Theorem 3.4.3. *The functions $\phi_{\tau, h}$ and $\phi_h^{(r)}$ have the vanishing property in the sense of Definition 2.1.6.*

As a consequence, by Lemma 2.5.4, both functions have base change transfers to $C_c^\infty(G(\mathbb{Q}_p))$. The next result provides a particular base change transfer with specified spectral information.

Theorem 3.4.4. *For any $h \in C_c^\infty(\mathcal{G}_{\mathcal{L}}(\mathbb{Z}_p))$, the function $z_{\tau, -\mu} * h \in C_c^\infty(G^*(\mathbb{Q}_p))$ is a base change transfer of $\phi_{\tau, h} \in C_c^\infty(G(\mathbb{Q}_{p^r}))$. Similarly, $z_{-\mu} * h$ is a base change transfer of $\phi_h^{(r)}$.*

These results will be proved by global methods in §5. Here we record the following consequence that will not be used in what follows but is of independent interest.

Corollary 3.4.5. *Suppose that $h \in C_c^\infty(\mathcal{G}_{\mathcal{L}}(\mathbb{Z}_p))$ is a base change transfer of a function $\tilde{h} \in C_c^\infty(G(\mathbb{Q}_{p^r}))$ that has the vanishing property. Then for any $\delta \in G(\mathbb{Q}_{p^r})$ such that $N_r \delta$ is semisimple, we have $TO_{\delta\sigma}(\phi_{\tau, h}) = TO_{\delta\sigma}(z_{\tau, -\mu}^{(r)} * \tilde{h})$ and $TO_{\delta\sigma}(\phi_h^{(r)}) = TO_{\delta\sigma}(z_{-\mu}^{(r)} * \tilde{h})$.*

Proof. By Theorem 3.4.3 and Proposition 2.5.5, both functions $\phi_{\tau, h}$ and $z_{\tau, -\mu}^{(r)} * \tilde{h}$ have the vanishing property. By Theorem 3.4.4 and Theorem 2.5.7, both functions have base change transfer $z_{\tau, -\mu} * h$, in the sense of Lemma 2.5.4. Thus the two functions have the same twisted orbital integrals. The identity for $\phi_h^{(r)}$ and $z_{-\mu}^{(r)} * \tilde{h}$ is proved similarly. $\square$

4. SIMPLE SHIMURA VARIETIES OF KOTTWITZ TYPE

In this section we prove our main global result (Theorem 4.3.1) on the description of the cohomology of the simple Shimura varieties studied by Kottwitz in [Kot92a]. Our result generalizes the main result from *loc. cit.* to many cases of bad reduction.

4.1. Basic setup. First we recall the basic setting from [Kot92a], cf. also [Hai05, §5.2].

4.1.1. Shimura data. Let $\mathbb{F}$ be a CM field with totally real subfield $\mathbb{F}_0$. Denote by $c \in \text{Gal}(\mathbb{F}/\mathbb{F}_0)$ the complex conjugation. Let $\mathbb{D}$ be a central division algebra of dimension n^2 over $\mathbb{F}$ and $*$ an anti-involution of the second kind on $\mathbb{D}$. Let $\mathbb{G}$ be the $\mathbb{Q}$ -group whose points in any commutative $\mathbb{Q}$ -algebra R are given by

$$\mathbb{G}(R) = \{x \in \mathbb{D} \otimes_{\mathbb{Q}} R \mid xx^* \in R^\times\}$$

The map $x \mapsto xx^*$ defines the similitude homomorphism $c : \mathbb{G} \rightarrow \mathbb{G}_m$ whose kernel $\mathbb{G}_0$ is an inner form of (the restriction of scalars of) a unitary group associated to the quadratic extension $\mathbb{F}/\mathbb{F}_0$.

Let $h_0 : \mathbb{C} \rightarrow \mathbb{D} \otimes_{\mathbb{Q}} \mathbb{R}$ be an $\mathbb{R}$ -algebra homomorphism such that $h_0(\bar{z}) = h_0(z)^*$ for all $z \in \mathbb{C}$. Assume moreover that $x \mapsto h_0(i)^{-1} x^* h_0(i)$ defines a positive involution on $\mathbb{D} \otimes_{\mathbb{Q}} \mathbb{R}$. Let $h : \text{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m \rightarrow \mathbb{G}_{\mathbb{R}}$ be the inverse of the restriction of h_0 to $\mathbb{C}^\times$ and let $X_\infty := \mathbb{G}(\mathbb{R})/K_\infty$ be the conjugacy class of h in $\mathbb{G}(\mathbb{R})$, where K_∞ is the centralizer of h in $\mathbb{G}(\mathbb{R})$. Fix an isomorphism $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C} \times \mathbb{C}$ where the first (resp. second) copy of $\mathbb{C}$ on the right hand side corresponds to the identity (resp. complex conjugation). Restricting $h_{\mathbb{C}}$ to the first factor, we obtain a cocharacter $\mu : \mathbb{G}_{m, \mathbb{C}} \rightarrow \mathbb{G}_{\mathbb{C}}$. Then the pair $(\mathbb{G}, X_\infty)$ is a Shimura datum. Let $\mathbb{E}$ be its reflex field, i.e. the field of definition of the $\mathbb{G}(\mathbb{C})$ -conjugacy class of μ . Then we get a tower of Shimura varieties Sh_K for neat compact open subgroups $K \subset \mathbb{G}(\mathbb{A}_f)$, each of which is a projective smooth algebraic variety over $\mathbb{E}$.

4.1.2. *PEL data.* The Shimura variety Sh_K (more precisely a finite disjoint union of copies of it) can be realized as a moduli space of abelian varieties with extra structures involving polarizations, endomorphisms and level structures. To achieve this we first define the corresponding PEL datum.

Let $\mathbb{B} := \mathbb{D}^{\mathrm{op}}$ and $\mathbb{V} = \mathbb{D}$ the left $\mathbb{B}$ -module via right multiplication. As explained in [Hai05, §5.2.1], we can find $\xi \in \mathbb{D}^\times$ such that $\xi^* = -\xi$ and the involution $\dagger$ on $\mathbb{B} = \mathbb{D}^{\mathrm{op}}$ defined by $x^\dagger := \xi x^* \xi^{-1}$ is positive. Then the pairing $(\cdot, \cdot) : \mathbb{V} \times \mathbb{V} \rightarrow \mathbb{Q}$ defined by

$$(x, y) := \mathrm{tr}_{\mathbb{D}/\mathbb{Q}}(x \xi y^*)$$

gives $\mathbb{V}$ the structure of a non-degenerate $\dagger$ -Hermitian $\mathbb{B}$ -module. Moreover, replacing ξ by $-\xi$ if necessary, we may assume that $(\cdot, h_0(i)\cdot)$ is positive definite.

The cocharacter μ induces a decomposition of $\mathbb{B}$ -modules $\mathbb{V}_{\mathbb{C}} = \mathbb{V}_1 \oplus \mathbb{V}_2$ where $\mu(z)$ acts by z^{-1} (resp. 1) on $\mathbb{V}_1$ (resp. $\mathbb{V}_2$) for all $z \in \mathbb{C}^\times$. The reflex field $\mathbb{E}$ is also the field of definition of the $\mathbb{B}$ -module $\mathbb{V}_1$.

Remark 4.1.1. Our convention on the cocharacter μ is consistent with [Kot92b], [Hai05, §5.2.2], [SS13] and is opposite to [RZ96], [Sch13b].

4.1.3. *Integral data.* Fix a prime p such that $\mathbb{F}/\mathbb{F}_0$ is split above any prime of $\mathbb{F}_0$ over p . Let $\mathfrak{p}$ be a prime of $\mathbb{E}$ over p and let $\mathbb{E}_{\mathfrak{p}}$ be the $\mathfrak{p}$ -adic completion of $\mathbb{E}$. To define integral models of the Shimura varieties over $\mathrm{Spec} \mathcal{O}_{\mathbb{E}_{\mathfrak{p}}}$ we fix integral PEL data as follows. Let $\mathcal{O}_{\mathbb{D}}$ be a maximal $\mathbb{Z}_{(p)}$ -order in $\mathbb{D}$ that is stable under the involution $\dagger$ and let $\mathcal{O}_{\mathbb{B}} := \mathcal{O}_{\mathbb{D}}^{\mathrm{op}}$. Denote $B = \mathbb{B} \otimes_{\mathbb{Q}} \mathbb{Q}_p$, $V = \mathbb{V} \otimes_{\mathbb{Q}} \mathbb{Q}_p$ and $\mathcal{O}_B := \mathcal{O}_{\mathbb{B}} \otimes_{\mathbb{Z}} \mathbb{Z}_p$. Then $\mathcal{O}_B$ is a maximal $\mathbb{Z}_p$ -order in B . Let $v_1, \dots, v_s$ be the places of $\mathbb{F}_0$ above p . For each $1 \leq i \leq s$, there are two places w_i, w_i^c of $\mathbb{F}$ above v_i . Let $B_i := \mathbb{B} \otimes_{\mathbb{F}} \mathbb{F}_{w_i}$, $V_i := \mathbb{V} \otimes_{\mathbb{F}} \mathbb{F}_{w_i}$ and $\mathcal{O}_{B_i} := \mathcal{O}_{\mathbb{B}} \otimes_{\mathcal{O}_{\mathbb{F}}} \mathcal{O}_{\mathbb{F}_{w_i}}$. Then the involution $\dagger$ induces isomorphisms $\mathcal{O}_{B_i}^{\mathrm{op}} \cong \mathcal{O}_{B_i} \otimes_{\mathcal{O}_{\mathbb{F}}} \mathcal{O}_{\mathbb{F}_{w_i^c}}$ and $B_i^{\mathrm{op}} \cong \mathbb{B} \otimes_{\mathbb{F}} \mathbb{F}_{w_i^c}$. The alternating form $(\cdot, \cdot)$ on $\mathbb{V}$ induces isomorphisms $\mathbb{V} \otimes_{\mathbb{F}} \mathbb{F}_{w_i^c} \cong V_i^*$ where V_i^* is the $\mathbb{Q}_p$ -linear dual of V_i . Consequently, we get isomorphisms

$$\mathbb{G}_0(\mathbb{Q}_p) \cong \prod_{i=1}^s B_i^\times, \quad \mathbb{G}(\mathbb{Q}_p) = \mathbb{G}_0(\mathbb{Q}_p) \times \mathbb{Q}_p^\times.$$

For each $1 \leq i \leq s$, let $\mathcal{L}_i = \{\Lambda_{i,j}, j \in \mathbb{Z}\}$ be an $\mathcal{O}_{B_i}$ -lattice chain in V_i where we label the lattices so that $\Lambda_{i,j} \subset \Lambda_{i,j+1}$ for each $j \in \mathbb{Z}$. We extend this to a self-dual multi-chain of $\mathcal{O}_B$ -lattices $\mathcal{L}$ in V as follows. For each i , we extend $\mathcal{L}_i$ to a $(\cdot, \cdot)$ -self dual lattice chain $\mathcal{L}_i^+$ in $V_i \oplus V_i^*$ using the method of [Hai05, §5.2.3]; this depends explicitly on the choice of the element ξ above. We let $\mathcal{L}$ be the set of $\mathcal{O}_B$ -lattices in $V = \prod_{i=1}^s (V_i \oplus V_i^*)$ of the form $\prod_{i=1}^s \Lambda_i^+$ where $\Lambda_i^+ \in \mathcal{L}_i^+$ for each $i = 1, \dots, s$.

The stabilizer $\mathbf{P}_{\mathcal{L}}$ of $\mathcal{L}$ in $\mathbb{G}_0(\mathbb{Q}_p)$ is a parahoric subgroup and $K_{\mathcal{L}} := \mathbf{P}_{\mathcal{L}} \times \mathbb{Z}_p^\times$ is a parahoric subgroup of $\mathbb{G}(\mathbb{Q}_p)$. The corresponding $\mathbb{Z}_p$ -model of $\mathbb{G}$ (resp. $\mathbb{G}_0$) will be denoted by $\mathbf{G}_{\mathcal{L}}$ (resp. $\mathbf{G}_{\mathcal{L},0}$). Finally we fix a neat open compact subgroup $K^p \subset \mathbb{G}(\mathbb{A}_f^p)$.

4.2. **Integral models.** We will define integral models of Shimura varieties with level $K = K^p K_{\mathcal{L}}$ structure.

For any scheme S , let $AV_{\mathcal{O}_{\mathbb{B}}}(S)$ be the $\mathbb{Z}_{(p)}$ -linear category in which:

- The objects are pairs (A, i) where A is an abelian scheme over S and $i : \mathcal{O}_{\mathbb{B}} \rightarrow \mathrm{End}(A) \otimes \mathbb{Z}_{(p)}$ is a $\mathbb{Z}_{(p)}$ -algebra homomorphism;
- The morphisms between two objects (A, i) and (A', i') form the $\mathbb{Z}_{(p)}$ -module $\mathrm{Hom}_{\mathcal{O}_{\mathbb{B}}}(A, A')$ consisting of elements in $\mathrm{Hom}(A, A') \otimes \mathbb{Z}_{(p)}$ that commute with $\mathcal{O}_{\mathbb{B}}$.

For each object (A, i) in $AV_{\mathcal{O}_{\mathbb{B}}}(S)$, let $A^\vee$ be the dual abelian scheme and $i^\vee(b) := i(b^\dagger)^\vee$ for all $b \in \mathcal{O}_{\mathbb{B}}$. Then $(A^\vee, i^\vee)$ is also an object in $AV_{\mathcal{O}_{\mathbb{B}}}(S)$. A *polarization* of (A, i) is an isomorphism $\lambda : A \rightarrow A^\vee$ in $AV_{\mathcal{O}_{\mathbb{B}}}(S)$ such that $n\lambda$ is induced by an ample line bundle on A for some positive integer n . In particular, λ induces the Rosati involution ι_λ on $\mathrm{End}(A) \otimes \mathbb{Z}_{(p)}$ and $i(b^\dagger) = i(b)^{\iota_\lambda}$ for all $b \in \mathcal{O}_{\mathbb{B}}$. Note that what we call a *polarization* is called *principal polarization* in [RZ96, §6].

To define the moduli problem, recall from [RZ96, Definition 6.5] that an $\mathcal{L}$ -set of abelian varieties $A_\bullet = \{A_\Lambda, \Lambda \in \mathcal{L}\}$ in $AV_{\mathcal{O}_{\mathbb{B}}}(S)$ is a functor from $\mathcal{L}$ to $AV_{\mathcal{O}_{\mathbb{B}}}(S)$ satisfying a periodicity condition

and a condition on the height of the isogenies among the A_Λ 's. For each $\mathcal{L}$ -set $A_\bullet$, there is a dual $\mathcal{L}$ -set $\tilde{A}_\bullet$ and we have the notion of a polarization $\lambda : A_\bullet \rightarrow \tilde{A}_\bullet$, cf. [RZ96, Definition 6.6].

We define the functor $\mathfrak{M}_{K_\mathcal{L}K^p}$ on the category of $\mathcal{O}_{\mathbb{E}_p}$ -schemes that associates to a test scheme S the set of isomorphism classes of the following objects:

- An $\mathcal{L}$ -set of abelian varieties $A_\bullet = (A_\bullet, i_\bullet)$ in $AV_{\mathcal{O}_\mathbb{B}}(S)$;
- A polarization λ of $A_\bullet$;
- A K^p -level structure on $A_\bullet$, i.e. a K^p -orbit of isomorphisms of skew-Hermitian $\mathbb{B}$ -modules

$$\bar{\eta} : H_1(A_\bullet, \mathbb{A}_f^p) \cong \mathbb{V} \otimes \mathbb{A}_f^p \pmod{K^p}$$

that respect the bilinear forms on both sides up to a constant in $(\mathbb{A}_f^p)^\times$. More precisely, we consider such isomorphisms $\bar{\eta} : H_1(A_{\bullet,s}, \mathbb{A}_f^p) \cong V_{\mathbb{A}_f^p}$ for every geometric point s of S , and we require that $\bar{\eta}$ is fixed by $\pi_1(S, s)$ (see [Kot92b, p. 390]).

Moreover we require $A_\bullet$ to satisfy the usual Kottwitz determinant condition, i.e., an equality of polynomial functions

$$\det(b; \text{Lie} A_\Lambda) = \det(b; \mathbb{V}_1)$$

for all $\Lambda \in \mathcal{L}$ and all $b \in \mathcal{O}_\mathbb{B}$.

Proposition 4.2.1. *The functor $\mathfrak{M}_{K_\mathcal{L}K^p}$ is representable by a projective scheme over $\text{Spec } \mathcal{O}_{\mathbb{E}_p}$ whose generic fiber is isomorphic to a disjoint union of $|\ker^1(\mathbb{Q}, \mathbb{G})|$ copies of $\text{Sh}_{K_\mathcal{L}K^p}$.*

Proof. Each lattice $\Lambda \in \mathcal{L}$ determines a maximal parahoric subgroup K_Λ whose corresponding moduli functor $\mathfrak{M}_{K_\Lambda K^p}$ parametrizes a single object in $AV_{\mathcal{O}_\mathbb{B}}(S)$ with polarization and level K^p -structure, satisfying the determinant condition. By [Sch13b, Theorem 5.2], $\mathfrak{M}_{K_\Lambda K^p}$ is representable by a projective scheme $\mathcal{O}_{\mathbb{E}_p}$. Note that $\mathfrak{M}_{K_\mathcal{L}K^p}$ embeds as a closed subscheme of a product of finitely many $\mathfrak{M}_{K_\Lambda K^p}$'s, hence is also a projective scheme over $\mathcal{O}_{\mathbb{E}_p}$. The description of the generic fibre of $\mathfrak{M}_{K_\mathcal{L}K^p}$ follows from [Kot92b, §8]. $\square$

4.3. Description of cohomology. Let ξ be an irreducible finite dimensional algebraic representation of $\mathbb{G}$ that is defined over a number field L . Fix a place λ of L above a rational prime $\ell \neq p$, then by the construction in [Kot92b, §6] we obtain ℓ -adic local systems $\mathcal{F}_{\xi,K}$ (resp., $\mathcal{F}_{\xi,K_\mathcal{L}K^p}$) on Sh_K (resp., the integral model $\mathfrak{M}_{K_\mathcal{L}K^p}$) together with Hecke correspondences among them. Taking cohomology in degree $i \geq 0$, we get $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{E}) \times \mathbb{G}(\mathbb{A}_f)$ -representations

$$H_\xi^i := \varinjlim_K H_{\text{et}}^i(\text{Sh}_K \otimes_{\mathbb{E}} \overline{\mathbb{Q}}, \mathcal{F}_{\xi,K})$$

where $\overline{\mathbb{Q}}$ is the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. Here $K \subset \mathbb{G}(\mathbb{A}_f)$ ranges over all neat compact open subgroups of the form $K = K_p K^p$, where $K_p \subset K_\mathcal{L}$ is a normal compact open subgroup. We also form the virtual $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{E}) \times \mathbb{G}(\mathbb{A}_f)$ -representation

$$H_\xi^* := \sum_i (-1)^i H_\xi^i$$

where the alternating sum is taken in the corresponding Grothendieck group. Then we have a decomposition

$$H_\xi^* = \bigoplus_{\pi_f} \pi_f \otimes \sigma_\xi(\pi_f)$$

where π_f ranges over irreducible admissible representations of $\mathbb{G}(\mathbb{A}_f)$ and $\sigma_\xi(\pi_f)$ is a virtual finite dimensional representation of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{E})$. Now we can state our global main result, which gives a nearly complete description of the restriction of $\sigma_\xi(\pi_f)$ to the local Weil group at $\mathfrak{p}$.

Theorem 4.3.1. *For any self-dual $\mathcal{O}_B$ -lattice chain $\mathcal{L}$ in V with associated parahoric subgroup $K_\mathcal{L} \subset \mathbb{G}(\mathbb{Q}_p)$ as in §4.1.3, we have an equality in the Grothendieck group of $\mathbb{G}(\mathbb{A}_f^p) \times K_\mathcal{L} \times W_{\mathbb{E}_p}$ representations*

$$H_\xi^* = \sum_{\pi_f} a(\pi_f) \pi_f \otimes (r_{-\mu} \circ \varphi_{\pi_p}|_{W_{\mathbb{E}_p}}) \cdot |\cdot|_{\mathbb{E}_p}^{-\dim \text{Sh}/2}$$

where $a(\pi_f) \in \mathbb{Z}$ is defined in [Kot92a, p.657].

Remark 4.3.2. As remarked in [SS13, §5], we expect this to be an equality in the Grothendieck group of $\mathbb{G}(\mathbb{A}_f) \times W_{\mathbb{E}_p}$ -representations. Under some extra assumptions the expectation is true. We refer to [SS13, Theorem 5.2, Corollary 5.3] for details.

The proof of Theorem 4.3.1 will be finished in §4.5, as a consequence of the local results Theorem 3.4.3 and Theorem 3.4.4.

First we record its implications for the semisimple local L -factors of the Shimura varieties Sh_K at $\mathfrak{p}$. We introduce more notations, following [Hai14, §6.3]. Let $r_{\mathfrak{p}} : \hat{G} \rtimes W_{\mathbb{Q}_p} \rightarrow \text{GL}(V_{r_{\mathfrak{p}}})$ be the algebraic representation defined by

$$r_{\mathfrak{p}} := \text{Ind}_{\hat{G} \rtimes W_{\mathbb{E}_p}}^{\hat{G} \rtimes W_{\mathbb{Q}_p}} r_{-\mu}$$

where $r_{-\mu} : \hat{G} \rtimes W_{\mathbb{E}_p} \rightarrow \text{GL}(V_{-\mu})$ is the representation defined in §3.4. For an irreducible smooth representation π_p of $\mathbb{G}(\mathbb{Q}_p) = G(\mathbb{Q}_p)$, let $\varphi_{\pi_p} : W_{\mathbb{Q}_p} \rightarrow \hat{G} \rtimes W_{\mathbb{Q}_p}$ be its *semisimple L -parameter*. Then the semisimple local L -factor of π_p with respect to the representation $r_{\mathfrak{p}}$ is defined by

$$L^{ss}(s, \pi_p, r_{\mathfrak{p}}) = \det(\text{Id} - p^{-s} r_{\mathfrak{p}}(\Phi_p); V_{r_{\mathfrak{p}}}^{I_{\mathbb{Q}_p}})$$

where $I_{\mathbb{Q}_p} \subset W_{\mathbb{Q}_p}$ is the inertia subgroup and $\Phi_p \in W_{\mathbb{Q}_p}$ is a geometric Frobenius element.

Corollary 4.3.3. *Let $K \subset \mathbb{G}(\mathbb{A}_f)$ be any sufficiently small compact open subgroup which factorizes in the form $K = K_p K^p$ specified above. Then the semisimple local Hasse-Weil factor of Sh_K at the place $\mathfrak{p}$ of $\mathbb{E}$ is given by*

$$\zeta_{\mathfrak{p}}^{ss}(\text{Sh}_K, s) = \prod_{\pi_f} L^{ss}(s - \frac{\dim \text{Sh}_K}{2}, \pi_p, r_{\mathfrak{p}})^{a(\pi_f) \dim \pi_f^K}.$$

Proof. We denote $E = \mathbb{E}_p$ and $d := \dim \text{Sh}_K$. Let E_0 be the maximal unramified extension of $\mathbb{Q}_p$ in E . Write $K = K_p K^p$ where $K^p \subset \mathbb{G}(\mathbb{A}_f^p)$ and K_p is contained in a parahoric subgroup $K_{\mathcal{L}}$ in $\mathbb{G}(\mathbb{Q}_p)$ associated to some $\mathcal{O}_B$ -stable lattice chain $\mathcal{L}$ in V . Recall from [Hai05, Definition 9.4] that

$$\log \zeta_{\mathfrak{p}}^{ss}(\text{Sh}_K, s) = \sum_{j=1}^{\infty} \text{Tr}^{ss}(\Phi_{\mathfrak{p}}^j, H^*(\text{Sh}_{K, \overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_{\ell})) \frac{N_{\mathfrak{p}}^{-js}}{j}$$

where the semisimple trace Tr^{ss} is (by definition) the trace of $\Phi_{\mathfrak{p}}^j$ on the I_E -invariant subspace of the *semisimplification* of the corresponding W_E -representations.

We take ξ to be the trivial representation and apply Theorem 4.3.1. Then for any $\tau \in \Phi_{\mathfrak{p}}^j I_E \subset W_E$, where $\Phi_{\mathfrak{p}} \in W_E$ is a geometric Frobenius element, we get that

$$\text{Tr}(\tau, H^*(\text{Sh}_{K, \overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_{\ell})) = \sum_{\pi_f} a(\pi_f) \dim(\pi_f^K) \text{Tr}(r_{-\mu} \circ \varphi_{\pi_p}(\tau))(N_{\mathfrak{p}})^{-jd/2}.$$

Integrating both sides over the coset $\tau \in \Phi_{\mathfrak{p}}^j I_E$ (against the Haar measure on I_E with total volume 1) and taking the sum over j we get

$$(4.1) \quad \log \zeta_{\mathfrak{p}}^{ss}(\text{Sh}_K, s - \frac{d}{2}) = \sum_{\pi_f} a(\pi_f) \dim(\pi_f^K) \sum_{j=1}^{\infty} \text{Tr}^{ss}(\varphi_{\pi_p}(\Phi_{\mathfrak{p}}^j), r_{-\mu}) \frac{(N_{\mathfrak{p}})^{-js}}{j}.$$

The semisimple trace on the right hand side of (4.1) coincides with the trace on the inertia invariant subspace since φ_{π_p} is the *semisimple L -parameter* so that the representation $r_{-\mu} \circ \varphi_{\pi_p}|_{W_E}$ is such that inertia acts through a finite quotient. In other words we have

$$\text{Tr}^{ss}(\varphi_{\pi_p}(\Phi_{\mathfrak{p}}^j), r_{-\mu}) = \text{Tr}(\varphi_{\pi_p}(\Phi_{\mathfrak{p}}^j)|_{r_{-\mu}^{I_E}}).$$

On the other hand, recall from [Hai05, Definition 9.8] that

$$\log L^{ss}(s, \pi_p, r_{\mathfrak{p}}) = \sum_{r=1}^{\infty} \text{Tr}^{ss}(\varphi_{\pi_p}(\Phi_p^r); V_{r_{\mathfrak{p}}}) \frac{p^{-rs}}{r}.$$

By [Hai14, Lemma 6.3.1] we have

$$\mathrm{Tr}^{ss}(\varphi_{\pi_p}(\Phi_p^r); V_{r_p}) = \begin{cases} [E_0 : \mathbb{Q}_p] \mathrm{Tr}^{ss}(\varphi_{\pi_p}(\Phi_p^j), r_{-\mu}^{E_0}) & \text{if } r = j[E_0 : \mathbb{Q}_p] \\ 0 & \text{if } [E_0 : \mathbb{Q}_p] \nmid r \end{cases}$$

where $r_{-\mu}^{E_0} := \mathrm{Ind}_{\hat{G} \rtimes W_E}^{\hat{G} \rtimes W_{E_0}} r_{-\mu}$. Since the extension E/E_0 is totally ramified, we have

$$I_{E_0}/I_E = \frac{\hat{G} \rtimes W_{E_0}}{\hat{G} \rtimes W_E}$$

from which we deduce that $(r_{-\mu}^{E_0})^{I_{E_0}} = r_{-\mu}^{I_E}$ and hence

$$\mathrm{Tr}^{ss}(\varphi_{\pi_p}(\Phi_p^j), r_{-\mu}^{E_0}) = \mathrm{Tr}(\varphi_{\pi_p}(\Phi_p^j), (r_{-\mu}^{E_0})^{I_{E_0}}) = \mathrm{Tr}(\varphi_{\pi_p}(\Phi_p^j), r_{-\mu}^{I_E}) = \mathrm{Tr}^{ss}(\varphi_{\pi_p}(\Phi_p^j), r_{-\mu}).$$

Therefore we get

$$(4.2) \quad \log L^{ss}(s, \pi_p, r_p) = \sum_{j=1}^{\infty} \mathrm{Tr}^{ss}(\varphi_{\pi_p}(\Phi_p^j), r_{-\mu}^{E_0}) \frac{p^{-j[E_0:\mathbb{Q}_p]s}}{j} = \sum_{j=1}^{\infty} \mathrm{Tr}^{ss}(\varphi_{\pi_p}(\Phi_p^j), r_{-\mu}) \frac{(N\mathfrak{p})^{-js}}{j}.$$

Now the result follows by comparing (4.1) and (4.2). $\square$

4.4. Counting points. Let $E = \mathbb{E}_p$. Fix an integer $j \geq 1$ and an element $\tau \in \mathrm{Frob}_E^j I_E \subset W_E$. Denote $r := j[\kappa_E : \mathbb{F}_p]$ where κ_E is the residue field of E and set $k_r := \mathbb{F}_{p^r}$, which is a degree j extension of κ_E . Let $h \in C_c^\infty(K_{\mathcal{L}})$ and $f^p \in C_c^\infty(\mathbb{G}(\mathbb{A}_f^p))$. Our goal is to calculate $\mathrm{Tr}(\tau \times h f^p | H_\xi^*)$ by the Grothendieck-Lefschetz trace formula. We mostly follow the exposition of [Sch13b]. Without loss of generality, we may assume that f^p is the characteristic function of a double coset $K^p g^p K^p$ divided by the volume of K^p for a neat compact open subgroup $K^p \subset \mathbb{G}(\mathbb{A}_f^p)$, and h is the characteristic function of $K_p g_p K_p$ divided by the volume of K_p for some normal open subgroup $K_p \subset K_{\mathcal{L}}$ and $g_p \in K_{\mathcal{L}}$.

4.4.1. Frobenius-Hecke correspondence. We will analyze the cohomological correspondence on $\mathrm{Sh}_{K_p K^p}$ that gives rise to the action of $\tau \times h f^p$ on H_ξ^* . The first step is to consider the induced correspondence on the “base level” $\mathrm{Sh}_{K_{\mathcal{L}} K^p}$, which extends to the integral model $\mathfrak{M}_{K_{\mathcal{L}} K^p}$, and describe its fixed points in the special fiber. More precisely the action of $h f^p$ is defined by a correspondence

$$(4.3) \quad \mathfrak{M}_{K_{\mathcal{L}} K^p} \xleftarrow{p_1} \mathfrak{M}_{K_{\mathcal{L}} K_{g_p}^p} \xrightarrow{p_2} \mathfrak{M}_{K_{\mathcal{L}} K^p}$$

where

- $K_{g_p}^p := K^p \cap g^p K^p (g^p)^{-1}$;
- p_1 is the composition of the natural projection $\mathfrak{M}_{K_{\mathcal{L}} K_{g_p}^p} \rightarrow \mathfrak{M}_{K_{\mathcal{L}}, g^p K^p (g^p)^{-1}}$ with the isomorphism $\mathfrak{M}_{K_{\mathcal{L}}, g^p K^p (g^p)^{-1}} \xrightarrow{\cong} \mathfrak{M}_{K_{\mathcal{L}} K^p}$ induced by (“right multiplication by”) g^p ;
- p_2 is the natural projection.

Our convention is slightly different from both [Sch13b] and [Kot92b].

4.4.2. Fixed points of correspondence. Let $\mathrm{Fix}_{j,\mathcal{L}}(g^p)$ denote the set of fixed points of $\tau \times h f^p$ in $\mathfrak{M}_{K_{\mathcal{L}} K_{g_p}^p}(\overline{\mathbb{F}}_p)$. The notation is meant to suggest that it is independent of $g_p \in K_{\mathcal{L}}$ and the choice of τ in the coset $\mathrm{Frob}_E^j I_E$. To describe it we consider the following correspondence of the special fiber

$$(4.4) \quad \mathfrak{M}_{K_{\mathcal{L}} K^p} \otimes \kappa_E \xleftarrow{\mathrm{Fr}_E^j \circ p_1} \mathfrak{M}_{K_{\mathcal{L}} K_{g_p}^p} \otimes \kappa_E \xrightarrow{p_2} \mathfrak{M}_{K_{\mathcal{L}} K^p} \otimes \kappa_E$$

where p_1, p_2 are the special fibers of the corresponding maps in (4.3) and Fr_E denotes the geometric Frobenius endomorphism of $\mathfrak{M}_{K_{\mathcal{L}} K^p} \otimes \kappa_E$. Then $\mathrm{Fix}_{j,\mathcal{L}}(g^p)$ is the fixed point set of the above correspondence, which consists of points $(A_\bullet, \lambda, \bar{\eta})$ in $\mathfrak{M}_{K_{\mathcal{L}} K_{g_p}^p}(\overline{\mathbb{F}}_p)$ such that $(A_\bullet, \lambda, \bar{\eta})$ and $\sigma^r(A_\bullet, \lambda, \bar{\eta} g^p)$ define the same point in $\mathfrak{M}_{K_{\mathcal{L}} K^p}(\overline{\mathbb{F}}_p)$. Here $\sigma^r(\cdot)$ means base change along p^r -th power map of $\overline{\mathbb{F}}_p$. Then for each $i \in \mathbb{Z}$, there exists an isomorphism $u_i : \sigma^r(A_i) \rightarrow A_i$ in $AV_{\mathcal{O}_{\mathbb{B}}}(\overline{\mathbb{F}}_p)$ sending $\sigma^r(\bar{\eta} g^p)$ to $\bar{\eta}$ and compatible with the isogenies among the A_i 's. Moreover, there is a number $c_0 \in \mathbb{Z}_{(p)}^\times \cap \mathbb{R}_{>0}$ such

that $u_0^* \lambda_0 = c_0 \sigma(\lambda_0)$. Then (A_0, u_0, λ_0) is a $p^r c_0$ -polarized virtual $\mathcal{O}_{\mathbb{B}}$ -abelian variety over $\mathbb{F}_{p^r}$ up to $\mathbb{Z}_{(p)}$ -isogeny in the sense of [Kot92b]. The inverses of the Frobenius endomorphisms on A_j define a conjugacy class $\gamma = (\gamma_l)_{l \neq p} \in \mathbb{G}(\mathbb{A}_f^p)$ and a σ -conjugacy class $\delta = (\delta_1, \delta_0) \in \mathbb{G}(\mathbb{Q}_{p^r}) = G(\mathbb{Q}_{p^r}) \times \mathbb{Q}_{p^r}^\times$ such that $\delta_0 \sigma(\delta_0) \cdots \sigma^{r-1}(\delta_0) = p^r c_0$.

Via u_i , the p -divisible groups $A_i[p^\infty]$ over $\overline{\mathbb{F}}_p$ descend to $\mathbb{F}_{p^r}$ and the resulting p -divisible $\mathcal{O}_{\mathbb{B}}$ -module decomposes according to the isomorphism $\mathcal{O}_{\mathbb{B}} \otimes \mathbb{Z}_p \cong \mathcal{O}_B \times \mathcal{O}_B^{op}$. Let H_i be the factor of $A_i[p^\infty]$ corresponding to $\mathcal{O}_B$. Then H_i is a p -divisible $\mathcal{O}_B$ -module over $\mathbb{F}_{p^r}$. The isogenies between A_i 's induce isogenies between H_i 's so that they form a p -divisible group with $(\mathcal{O}_B, \mu_1, \mathcal{L})$ -structure over $\mathbb{F}_{p^r}$, where $\mathbb{F}_{p^r}$ is given the structure of an $\mathcal{O}_E$ -algebra by viewing it as a degree j extension of the residue field κ_E . Here μ_1 is the coweight of $\mathbb{G}_{0, \overline{\mathbb{Q}}_p}$ defined as follows: we choose a field isomorphism $\overline{\mathbb{Q}}_p \cong \mathbb{C}$ and interpret μ as a coweight of $\mathbb{G}_{\overline{\mathbb{Q}}_p}$. Then μ_1 is the first factor of μ under the canonical isomorphism $\mathbb{G}_{\overline{\mathbb{Q}}_p} \cong \mathbb{G}_{0, \overline{\mathbb{Q}}_p} \times \mathbb{G}_{m, \overline{\mathbb{Q}}_p}$.

4.4.3. Fixed points in an isogeny class. Let (A, u, λ) be a $p^r c_0$ -polarized virtual $\mathcal{O}_{\mathbb{B}}$ -abelian variety over $\mathbb{F}_{p^r}$. Let $I_{(A, u, \lambda)}$ be the $\mathbb{Q}$ -algebraic group of self quasi-isogenies of A compatible with u, λ .

Let $\text{Isog}(A, u, \lambda)$ be the subset of $\text{Fix}_{j, \mathcal{L}}(g^p)$ consisting of points $(A_\bullet, \lambda, \bar{\eta})$ lying in the closure of the generic fiber³ whose associated polarized virtual $\mathcal{O}_{\mathbb{B}}$ -abelian variety (described as above) is isomorphic to (A, u, λ) .

Proposition 4.4.1. *There is an injection*

$$\text{Isog}(A, u, \lambda) \hookrightarrow I_{(A, u, \lambda)}(\mathbb{Q}) \backslash (Y_p \times Y^p)$$

where

- $Y_p := \{g \in \mathbb{G}(\mathbb{Q}_{p^r}) / K_{\mathcal{L}, r} \mid pg\Lambda \subset p\delta\sigma g\Lambda \subset g\Lambda, \forall \Lambda \in \mathcal{L}\}$
- $Y^p := \{x \in \mathbb{G}(\mathbb{A}_f^p) / K_{g^p}^p \mid x^{-1}\gamma x \in g^p K^p\}.$

Proof. The proof is similar to [Sch13b, Proposition 6.5], but we take into account our slightly different convention for the Frobenius-Hecke correspondence. A fixed point for the Frobenius-Hecke correspondence is a point $(A_\bullet, i_\bullet, \lambda, \bar{\eta}) =: (A_\bullet, \lambda, \bar{\eta}) \in \mathfrak{M}_{K_{\mathcal{L}} K_{g^p}^p}(\overline{\mathbb{F}}_p)$ such that there is an isomorphism $(A_\bullet, \lambda, \bar{\eta}) \cong \sigma^r(A_\bullet, \lambda, \bar{\eta} g^p)$ in $\mathfrak{M}_{K_{\mathcal{L}} K^p}(\overline{\mathbb{F}}_p)$. The condition that $\sigma^r(\bar{\eta} g^p) = \bar{\eta}$ as K^p -level structures translates to the condition $\gamma^{-1} x g^p K^p = x K^p$, or $x^{-1} \gamma x \in g^p K^p$. The rest of the argument is as in [Sch13b, Proposition 6.5]. $\square$

Corollary 4.4.2. *We have the following equality*

$$\text{Tr}(\tau \times h f^p | H_\xi^*) = \sum_{(A, u, \lambda)} \text{vol}(I(\mathbb{Q}) \backslash I(\mathbb{A}_f)) O_\gamma(f^p) T O_{\delta\sigma}(\phi_{\tau, h}) \text{tr } \xi(\gamma_\ell)$$

where the sum runs over isogeny classes of polarized virtual $\mathcal{O}_{\mathbb{B}}$ -abelian varieties (A, u, λ) over $\mathbb{F}_{p^r}$ and $I = I_{(A, u, \lambda)}$ is the automorphism group of (A, u, λ) .

Proof. Let $\pi : \text{Sh}_{K_p K^p} \rightarrow \text{Sh}_{K_{\mathcal{L}} K^p}$ be the natural projection. By the proper base change theorem, we have

$$H^*(\text{Sh}_{K_p K^p} \otimes_{\mathbb{E}} \overline{\mathbb{E}}, \mathcal{F}_\xi) \cong H^*(\text{Sh}_{K_{\mathcal{L}} K^p} \otimes_{\mathbb{E}} \overline{\mathbb{E}}, \pi_* \mathcal{F}_\xi) \cong H^*(\mathfrak{M}_{K_{\mathcal{L}} K^p} \otimes_{\kappa_E} \overline{\mathbb{F}}_p, R\psi \pi_* \mathcal{F}_\xi).$$

Under this isomorphism, the action of $\tau \times h f^p$ is defined from the composition

$$(\text{Fr}_E^j \circ p_1)^* R\psi \pi_* \mathcal{F}_\xi = p_1^*(\text{Fr}_E^j)^* R\psi \pi_* \mathcal{F}_\xi \xrightarrow{\tau} p_1^* R\psi \pi_* \mathcal{F}_\xi \xrightarrow{g_p g^p} p_2^* R\psi \pi_* \mathcal{F}_\xi$$

By [Sch13b, Proposition 5.5], there is a canonical isomorphism $R\psi \pi_* \mathcal{F}_\xi \cong \mathcal{F}_\xi \otimes R\psi \pi_* \mathbb{Q}_\ell$, under which the second map above factorizes as the tensor product of the cohomological correspondence on $\mathcal{F}_\xi$ induced by g^p and the cohomological correspondence on $R\psi \pi_* \mathbb{Q}_\ell$ induced by g_p . For a more thorough discussion of these points, we refer the reader to the forthcoming work of the second author with Rong Zhou and Yihang Zhu [HZZ].

³Note that the results of Pappas-Rapoport [PR03] show that the parahoric-level integral models we use, sometimes called naive, need not be flat.

Then the (naive) local trace of $\tau \times hf^p$ on $R\psi\pi_*\mathcal{F}_\xi$ at a fixed point $x = (A_\bullet, \lambda, \bar{\eta})$ is given by

$$(4.5) \quad \mathrm{Tr}(\tau \times hf^p | (R\psi\pi_*\mathcal{F}_\xi)_x) = \mathrm{Tr}(\xi(\gamma_\ell)) \mathrm{Tr}(\tau \times hf^p | (R\psi\pi_*\mathbb{Q}_\ell)_x).$$

By the version of the Lefschetz trace formula in [Var07, Theorem 2.3.2 (b)] and (4.5), we get

$$\mathrm{Tr}(\tau \times hf^p | H_\xi^*) = \sum_{x \in \mathrm{Fix}_{j,\mathcal{L}}(g^p)} \mathrm{Tr}(\tau \times hf^p | (R\psi\pi_*\mathcal{F}_\xi)_x).$$

We break the sum according to the partition $\mathrm{Fix}_{j,\mathcal{L}}(g^p) = \bigsqcup \mathrm{Isog}(A, u, \lambda)$ into isogeny classes labeled by isomorphism classes of polarized virtual $\mathcal{O}_\mathbb{B}$ -abelian varieties (A, u, λ) over $\mathbb{F}_{p^r}$.

Following the discussion in [Sch13b, §6.6], The contribution of $\mathrm{Isog}(A, u, \lambda)$ to the summation above is equal to

$$\mathrm{vol}(I(\mathbb{Q}) \backslash I(\mathbb{A}_f)) O_\gamma(f^p) TO_{\delta\sigma}(\phi_{\tau,h}) \mathrm{tr} \xi(\gamma_\ell).$$

Indeed, this follows from (4.5) and Proposition 4.4.1 by noting that if $(g, x) \in Y_p \times Y^p$ satisfies

$$\phi_{\tau,h}(g^{-1}\delta\sigma(g)) \neq 0,$$

then (g, x) lies in $\mathrm{Isog}(A, u, \lambda)$. Let us explain this last assertion. We denote by $(A_\bullet, \lambda, \bar{\eta})$ a point in $\mathrm{Fix}_{j,\mathcal{L}}(g^p)$ giving rise to the isogeny class (A, u, λ) and the pair (γ, δ) ; we may assume that it maps to the pair $(1, 1) \in Y_p \times Y^p$ under the injection of Proposition 4.4.1. Now the data (g, x) such that $\phi_{\tau,h}(g^{-1}\delta\sigma(g)) \neq 0$ give us an $\mathcal{L}$ -set of polarized abelian varieties $A'_\bullet$, and a quasi-isogeny $A_\bullet \rightarrow A'_\bullet$ which we use to transport the $\mathcal{O}_\mathbb{B}$ -action i on $A_\bullet$ to an action i' on $A'_\bullet$. Similarly, the $K_{g^p}^p$ -level structure is transported using x . By construction we get $(A'_\bullet, \lambda', \bar{\eta}')$. Once we know it satisfies the Kottwitz determinant condition, we will know it is a point in our moduli problem, and thus a point in $\mathrm{Isog}(A, u, \lambda)$, which maps to (g, x) under the injection of Proposition 4.4.1. But in this situation, the determinant condition is reflected exactly by the relative position of the lattices required in the definition of Y_p , as explained in [Hai05, §11.1.1]. $\square$

4.5. Pseudo-stabilization.

4.5.1. *Kottwitz triples.* To proceed further, we first recall the notion of Kottwitz triple as stated in [Sch13b, Definition 5.6].

Definition 4.5.1. Let $j \in \mathbb{Z}_{\geq 1}$ and set $r := j[\kappa_{\mathbb{E}_p} : \mathbb{F}_p]$. A degree j Kottwitz triple $(\gamma_0; \gamma, \delta)$ consists of

- A semisimple stable conjugacy class $\gamma_0 \in \mathbb{G}(\mathbb{Q})$ that is elliptic in $\mathbb{G}(\mathbb{R})$,
- a conjugacy class $\gamma \in \mathbb{G}(\mathbb{A}_f^p)$ that is stably conjugate to γ_0 ,
- a σ -conjugacy class $\delta \in \mathbb{G}(\mathbb{Q}_{p^r})$ whose naive norm $N_r\delta = \delta\sigma(\delta) \cdots \sigma^{r-1}(\delta)$ is stably conjugate to γ_0 , such that $\kappa_{\mathbb{G}_{\mathbb{Q}_p}}(p\delta) = \mu^\#$ in $X^*(Z(\hat{\mathbb{G}})^{\Gamma(p)})$, where $\Gamma(p) := \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$.

Proposition 4.5.2. Let $x \in \mathrm{Fix}_{j,\mathcal{L}}(g^p)$ with associated polarized virtual $\mathcal{O}_\mathbb{B}$ -abelian variety (A, u, λ) over $\mathbb{F}_{p^r}$, giving rise to conjugacy class $\gamma \in \mathbb{G}(\mathbb{A}_f^p)$ and σ -conjugacy class $\delta \in \mathbb{G}(\mathbb{Q}_{p^r})$. Suppose that the naive norm $N_r\delta = \delta\sigma(\delta) \cdots \sigma^{r-1}(\delta) \in \mathbb{G}(\mathbb{Q}_{p^r})$ is $\mathbb{G}(\mathbb{Q}_{p^r})$ -conjugate to an element in $\mathbb{G}(\mathbb{Q}_p)$. Then there exists a unique semisimple conjugacy class $\gamma_0 \in \mathbb{G}(\mathbb{Q})$ such that $(\gamma_0, \gamma, \delta)$ is a degree j -Kottwitz triple.

Proof. This is claimed in [She18, Lemma 5.3], but the proof given there contains a gap, and in fact does not use the strong hypothesis on δ . We will prove this by generalizing the argument of Kottwitz in [Kot92b, §14]. Let I be the reductive $\mathbb{Q}$ -group of automorphisms of (A, u, λ) . Kottwitz chose an arbitrary maximal $\mathbb{Q}$ torus $T \subset I$ and showed that it can be embedded in G , and he then took γ_0 to be the inverse of the Frobenius endomorphism π_A . To make the Kottwitz argument work in our situation, we make the further assumption that $T_{\mathbb{Q}_p}$ is elliptic in $I_{\mathbb{Q}_p}$. One can see that such $\mathbb{Q}$ -tori exist by [KPS22, Lemma 1.2.2] (and the references there). The assumption that the conjugacy class of $N_r\delta$ contains an element defined over $\mathbb{Q}_p$ implies that $I_{\mathbb{Q}_p}$ is an inner form of a subgroup of $G_{\mathbb{Q}_p}$. Then we use the fact that an elliptic torus in $I_{\mathbb{Q}_p}$ transfers to any of its inner forms to remove the assumption in [Kot92b, §14] that $G_{\mathbb{Q}_p}$ is quasi-split. More precisely, this assumption is used at two

places: first on p.420, paragraph above Lemma 14.1, to show that the centralizer N of T in $\text{End}_B(A)$ can be embedded in C locally at p ; then on p.421 where one shows that T_0 transfers to A_α at p .

Let us elaborate more on the first point. We let $\tilde{I} = \text{End}_B(A)^\times$ (resp., $\tilde{G} = C^\times$) be the $\mathbb{Q}$ -group of units of the semisimple algebra $\text{End}_B(A)$ (resp., $C = \text{End}_B(V)$). Then I (resp., G) is a closed subgroup of $\tilde{I}$ (resp., $\tilde{G}$). By assumption there is an element $\gamma_p \in G(\mathbb{Q}_p)$ that is stably conjugate to $N_r\delta$. Then the centralizer G_{γ_p} (resp., $\tilde{G}_{\gamma_p}$) is an inner form of $I_{\mathbb{Q}_p} = G_{\delta\sigma}$ (resp., $\tilde{I}_{\mathbb{Q}_p} = \tilde{G}_{\delta\sigma}$). Moreover, the class of $\tilde{G}_{\gamma_p}$ in $H^1(\mathbb{Q}_p, \tilde{I}_{\text{ad}})$ is equal to the image of the class of G_{γ_p} under the natural map $H^1(\mathbb{Q}_p, I_{\text{ad}, \mathbb{Q}_p}) \rightarrow H^1(\mathbb{Q}_p, \tilde{I}_{\text{ad}, \mathbb{Q}_p})$; see the proof of [Kot82, Lem. 5.8]. Let $\tilde{T} = \text{Res}_{N/\mathbb{Q}} \mathbb{G}_m$ be the centralizer of T in $\tilde{I}$. Let T_{ad} (resp., $\tilde{T}_{\text{ad}}$) be the image of T (resp., $\tilde{T}$) in the adjoint groups I_{ad} (resp., $\tilde{I}_{\text{ad}}$). Then we have a commutative diagram

$$\begin{array}{ccc} H^1(\mathbb{Q}_p, T_{\text{ad}}) & \longrightarrow & H^1(\mathbb{Q}_p, I_{\text{ad}}) \\ \downarrow & & \downarrow \\ H^1(\mathbb{Q}_p, \tilde{T}_{\text{ad}}) & \longrightarrow & H^1(\mathbb{Q}_p, \tilde{I}_{\text{ad}}). \end{array}$$

Since $T_{\mathbb{Q}_p}$ is elliptic in $I_{\mathbb{Q}_p}$, it transfers to the inner form G_{γ_p} and hence the class of G_{γ_p} lies in the image of the upper horizontal arrow. From the commutativity of the diagram we see that the class of $\tilde{G}_{\gamma_p}$ lies in the image of the lower horizontal arrow, which means that $\tilde{T}_{\mathbb{Q}_p}$ transfers to the inner form $\tilde{G}_{\mathbb{Q}_p} = C_{\mathbb{Q}_p}^\times$.

Now we get an embedding $\tilde{T}_{\mathbb{Q}_p} \rightarrow \tilde{G}_{\mathbb{Q}_p}$ which on $\mathbb{Q}_p$ -points gives rise to a group homomorphism $\varphi : N_{\mathbb{Q}}^\times \rightarrow C_{\mathbb{Q}_p}^\times$. We want to show that φ extends to an embedding of $\mathbb{Q}_p$ -algebras $N_{\mathbb{Q}_p} \rightarrow C_{\mathbb{Q}_p}$. Recall that N is a finite product of fields (cf. [Kot92b, p.420]). Take an element $\gamma \in N_{\mathbb{Q}_p}^\times$ such that $N_{\mathbb{Q}_p} = \mathbb{Q}_p[\gamma]$, and note that γ is automatically a regular semisimple element in $\tilde{I}(\mathbb{Q}_p)$: its centralizer in $\tilde{I}$ is precisely the maximal torus $\tilde{T}$, thanks to the double centralizer theorem applied to N in $\text{End}_B(A)$. Let $f_\gamma(X) \in \mathbb{Q}_p[X]$ be the characteristic polynomial of γ , viewed as an element in the semisimple $\mathbb{Q}_p$ -algebra $\text{End}_B(A)_{\mathbb{Q}_p}$. Then $f_\gamma(X)$ is also the minimal polynomial of γ and we have $N_{\mathbb{Q}_p} = \mathbb{Q}_p[X]/(f_\gamma(X))$. By the definition of transfer of semisimple elements, $f_\gamma(X)$ is the characteristic polynomial of $\varphi(\gamma) \in C_{\mathbb{Q}_p}$ and hence $f_\gamma(\varphi(\gamma)) = 0$ by the Cayley-Hamilton theorem. Thus φ extends to an embedding of $\mathbb{Q}_p$ -algebras $N_{\mathbb{Q}_p} = \mathbb{Q}_p[X]/(f_\gamma(X)) \rightarrow C_{\mathbb{Q}_p}$ and we are done. $\square$

Conversely, the construction in [Kot92b, §18] shows that every Kottwitz triple comes from a polarized virtual $\mathcal{O}_{\mathbb{B}}$ -abelian variety, keeping in mind that in our situation the invariant $\alpha(\gamma_0; \gamma, \delta)$ is automatically trivial (see [Kot92a]; one checks that the arguments in [Kot92b, §18] go through without the assumption made in [Kot92b, §5] that $\mathbb{B}_{\mathbb{Q}_p}$ is a product of matrix algebras over unramified extension of $\mathbb{Q}_p$). Altogether, combined with the vanishing property Theorem 3.4.4 we can follow the procedure in [Kot92b, §19] and convert the sum in Corollary 4.4.2 to a sum over equivalence classes of degree j Kottwitz triples:

$$(4.6) \quad \text{Tr}(\tau \times hf^p | H_\xi^*) = \sum_{(\gamma_0; \gamma, \delta)} c(\gamma_0; \gamma, \delta) O_\gamma(f^p) T O_{\delta\sigma}(\phi_{\tau, h}) \text{tr } \xi(\gamma_0)$$

where $c(\gamma_0; \gamma, \delta)$ is the product of $\text{vol}(I(\mathbb{Q}) \backslash I(\mathbb{A}_f))$ and the order of the finite group

$$\ker[\ker^1(\mathbb{Q}, I_0) \rightarrow \ker^1(\mathbb{Q}, \mathbb{G})].$$

Note that we do not need to multiply the right hand side of (4.6) by the order of $\ker^1(\mathbb{Q}, \mathbb{G})$, as this group is trivial for our choice of $\mathbb{G}$. (See, for example [Sch13a, proof of Proposition 3.2].) By the discussion in [Kot90, §4] and the fact that in our situation $|\mathfrak{K}(I_0/\mathbb{Q})| = 1$ in the notation of *loc.cit.* (cf. [Kot92a]), we have

$$(4.7) \quad c(\gamma_0; \gamma, \delta) = \tau(\mathbb{G}) \text{vol}(A_{\mathbb{G}}(\mathbb{R})^0 \backslash I(\mathbb{R}))^{-1}.$$

4.5.2. *Pseudo-coefficients.* Recall from [Kot92a] that the algebraic representation ξ of $\mathbb{G}(\mathbb{C})$ determines a discrete series L -packet of $\mathbb{G}(\mathbb{R})$. We let $f_{\xi,\infty}^{\mathbb{G}}$ be $(-1)^{q(\mathbb{G})}$ times the sum of pseudo-coefficients for representations in this L -packet, divided by the cardinality of the L -packet. Here $q(\mathbb{G})$ is half the real dimension of the symmetric space associated to $\mathbb{G}(\mathbb{R})$. By [Kot92a, Lemma 3.1], for all semisimple elements $\gamma_\infty \in \mathbb{G}(\mathbb{R})$, we have $SO_{\gamma_\infty}(f_{\xi,\infty}^{\mathbb{G}}) = 0$ unless γ_∞ is elliptic, in which case

$$(4.8) \quad SO_{\gamma_\infty}(f_{\xi,\infty}^{\mathbb{G}}) = \text{tr} \xi(\gamma_\infty) \text{vol}(A_{\mathbb{G}}(\mathbb{R})^0 \backslash I(\mathbb{R}))^{-1} e(I_{\gamma_\infty}^c)$$

where $I_{\gamma_\infty}^c$ is the inner form of the centralizer of γ_∞ in $\mathbb{G}(\mathbb{R})$ that is anisotropic modulo the center of $\mathbb{G}$ and $e(I_{\gamma_\infty}^c)$ is its Kottwitz sign whose definition we recall next. Actually we have $I_{\gamma_\infty}^c = I_{\mathbb{R}}$. Indeed, $I_{\mathbb{R}}$ is an inner form of $\mathbb{G}_{\mathbb{R},\gamma_0}$ (see [Kot92b, p. 423]) and we point out below that $I_{\mathbb{R}}$ is anisotropic mod the center of $G_{\mathbb{R}}$.

4.5.3. *Kottwitz sign.* In [Kot83], Kottwitz defines a sign $e(H) \in \{\pm 1\}$ for any connected reductive group H over a local field, which equals to 1 if H is quasi-split. Moreover, by the main theorem of *loc. cit.*, if H is a reductive group over a global field F , then there is a product formula $\prod_v e(H_v) = 1$, where the product is over all places of F .

In our situation, for any Kottwitz triple $(\gamma_0; \gamma, \delta)$ there is a reductive group I over $\mathbb{Q}$ such that

- for any prime $l \neq p$, $I_l \cong \mathbb{G}_{\gamma_l}$, the centralizer of γ_l in $\mathbb{G}(\mathbb{Q}_l)$;
- $I_p \cong \mathbb{G}_{\delta\sigma}$, the σ -centralizer of δ in $\mathbb{G}(\mathbb{Q}_{p^r})$;
- $I_{\mathbb{R}}$ is anisotropic mod center.

The existence of I is proved in [Kot90, §2] (the condition $\alpha(\gamma_0; \gamma, \delta) = 1$ in *loc. cit.* is automatic in our situation, cf. [Kot92a]). In fact, I is the automorphism group of a polarized virtual $\mathcal{O}_{\mathbb{B}}$ -abelian variety whose Kottwitz triple is equivalent to $(\gamma_0; \gamma, \delta)$. Let $e(\gamma) := \prod_{l \neq p} e(\mathbb{G}_{\gamma_l})$ (product over all finite primes different from p) and $e(\delta) := e(\mathbb{G}_{\delta\sigma})$. Then by the product formula, we have

$$e(\gamma)e(\delta) = e(I_{\mathbb{R}}).$$

4.5.4. *Finishing the proof of Theorem 4.3.1.* By Theorem 3.4.4 we have

$$(4.9) \quad e(\delta) TO_{\delta\sigma}(\phi_{\tau,h}) = SO_{\gamma_p}(f_{\tau,h})$$

where $\gamma_p \in \mathbb{G}(\mathbb{Q}_p)$ is stably conjugate to $\mathcal{N}\delta = \delta\sigma(\delta) \cdots \sigma^{r-1}(\delta)$ and $f_{\tau,h} := z_{\tau,-\mu} * h$.

Combining all the discussions above we get

$$\text{Tr}(\tau \times h f^p | H_{\xi}^*) = \tau(\mathbb{G}) \sum_{\gamma_0} SO_{\gamma_0}(f_{\tau,h} f^p f_{\xi,\infty}^{\mathbb{G}})$$

where the sum runs over *stable* conjugacy classes in $\mathbb{G}(\mathbb{Q})$ (which are automatically semisimple since $\mathbb{G}$ is anisotropic mod center). Then by [Kot92a, Lemma 4.1] and the simple trace formula for $\mathbb{G}$ we get

$$(4.10) \quad \begin{aligned} \text{Tr}(\tau \times h f^p | H_{\xi}^*) &= \sum_{\pi} m(\pi) \text{tr} \pi(f_{\tau,h} f^p f_{\xi,\infty}^{\mathbb{G}}) = \sum_{\pi_f} a(\pi_f) \text{tr} \pi_f(f_{\tau,h} f^p) \\ &= \sum_{\pi_f} a(\pi_f) \text{tr}(\tau | (r_{-\mu} \circ \varphi_{\pi_p} |_{W_{\mathbb{E}_p}})) \cdot | \cdot |^{-\dim \text{Sh}/2} \text{tr}(h f^p | \pi_f) \end{aligned}$$

where the first sum runs over automorphic representations π of $\mathbb{G}$, $m(\pi)$ denotes the automorphic multiplicity, while the second and third sums run over irreducible admissible representations of $\mathbb{G}(\mathbb{A}_f)$. The second equality uses the definition of $a(\pi_f)$ from [Kot92a, p. 657]:

$$(4.11) \quad a(\pi_f) = \sum_{\pi_\infty} m(\pi_f \otimes \pi_\infty) \text{tr} \pi_\infty(f_{\xi,\infty}^{\mathbb{G}}).$$

The third equality uses the definition of the element $z_{\tau,-\mu}$ in the stable Bernstein center of $\mathbb{G}(\mathbb{Q}_p)$. This finishes the proof of Theorem 4.3.1.

4.6. Lefschetz number via central functions. In this section we record some results of independent interest. We express the Lefschetz number in (4.6) and its semisimple variant in terms of elements in the Bernstein center instead of the function $\phi_{\tau,h}$.

As usual, we fix an integer $j \geq 1$, an element $\tau \in \text{Frob}_E^j I_E \subset W_E$ and let $r = j[\kappa_E : \mathbb{F}_p]$. Recall from §3.4 the element $z_{\tau,-\mu}^{(r)}$ in the (stable) Bernstein center of $\mathbb{G}(\mathbb{Q}_{p^r})$.

Proposition 4.6.1. *Suppose that $h \in C_c^\infty(\mathbf{G}_{\mathcal{L}}(\mathbb{Z}_p))$ is a base change transfer of a function $\tilde{h} \in C_c^\infty(\mathbb{G}(\mathbb{Q}_{p^r}))$ that satisfies the vanishing property in the sense of Definition 2.1.6. Then we have the following formula for the Lefschetz number*

$$\text{Tr}(\tau \times h f^p | H_{\xi}^*) = \sum_{(\gamma_0; \gamma, \delta)} c(\gamma_0; \gamma, \delta) O_{\gamma}(f^p) TO_{\delta\sigma}(z_{\tau,-\mu}^{(r)} * \tilde{h}) \text{tr} \xi(\gamma_0)$$

where the sum runs over equivalence classes of degree j -Kottwitz triples.

Proof. This is an immediate consequence of Equation (4.6) and Corollary 3.4.5. $\square$

In [Kot86a], for certain pairs of compact open subgroups $K_p \subset \mathbb{G}(\mathbb{Q}_p)$, $K_{p^r} \subset \mathbb{G}(\mathbb{Q}_{p^r})$ satisfying the axioms in *loc. cit.* p.240, Kottwitz shows that the idempotent $e_{K_p} = \frac{1}{\text{vol}(K_p)} 1_{K_p}$ is a base change transfer of the idempotent $e_{K_{p^r}} = \frac{1}{\text{vol}(K_{p^r})} 1_{K_{p^r}}$ and that the latter has the vanishing property (see the discussion on *loc. cit.* p.244, the paragraph above §2). For example, if $\mathcal{G}$ is a smooth connected group scheme over $\mathbb{Z}_p$ with generic fibre $\mathbb{G}_{\mathbb{Q}_p}$, then $K_p = \mathcal{G}(\mathbb{Z}_p)$ and $K_{p^r} = \mathcal{G}(\mathbb{Z}_{p^r})$ satisfy the axioms of Kottwitz.

The following consequence verifies a conjecture of the second author and Kottwitz (cf. [Hai14, Conjecture 6.1.1]) in our situation. It involves the element $z_{-\mu}^{(r)} = \int_{\text{Frob}_E^j I_E} z_{\tau,-\mu}^{(r)} d\tau$ in the (stable) Bernstein center of $\mathbb{G}(\mathbb{Q}_{p^r})$ defined in §3.4.

Corollary 4.6.2. *Let $K_p \subset \mathbb{G}(\mathbb{Q}_p)$ and $K_{p^r} \subset \mathbb{G}(\mathbb{Q}_p)$ be compact open subgroups satisfying Kottwitz's axiom as above. Let $K^p \subset \mathbb{G}(\mathbb{A}_f^p)$ be a sufficiently small compact open subgroup and $f^p \in C_c^\infty(K^p \backslash \mathbb{G}(\mathbb{A}_f^p) / K^p)$ a test function bi-invariant under K^p . Then we have*

$$\text{Tr}^{\text{ss}}(\text{Frob}_p^j \times f^p | H^*(\text{Sh}_{K_p K^p} \otimes_{\mathbb{E}} \bar{\mathbb{Q}}, \mathcal{F}_{K^p K_p})) = \sum_{(\gamma_0; \gamma, \delta)} c(\gamma_0; \gamma, \delta) O_{\gamma}(f^p) TO_{\delta\sigma}(z_{-\mu}^{(r)} * e_{K_{p^r}}) \text{tr} \xi(\gamma_0)$$

where the sum runs over equivalence classes of degree j -Kottwitz triples.

Proof. This follows by integrating the equation in the previous Proposition 4.6.1 over $\tau \in \text{Frob}_E^j I_E$ and taking $h = e_{K_p}$ (so that $\tilde{h}$ can be taken to be $e_{K_{p^r}}$ by Kottwitz's result in [Kot86a]). $\square$

5. PROOF OF THE LOCAL RESULTS

In this section we prove Theorem 3.4.3 and Theorem 3.4.4 by a local-global argument. It suffices to prove the local result for the group $G = \text{Res}_{F/\mathbb{Q}_p} \text{GL}_m(D)$ where F is a finite extension of $\mathbb{Q}_p$ and D a division algebra with center F . Then G is an inner form of the quasi-split group $G^* = \text{Res}_{F/\mathbb{Q}_p} \text{GL}_n$ where $n = m(\dim_F D)^{\frac{1}{2}}$. By Proposition 3.3.3, we may assume $B := M_m(D)^{\text{op}}$ and $V = M_m(D)$ viewed as a left B -module via right multiplication. Then $C \cong M_m(D)$ acts on V by left multiplication.

Let $\mathcal{O}_D$ be the unique maximal order in D and take $\mathcal{O}_B := M_m(\mathcal{O}_D)^{\text{op}}$. As in §3.1 we are given a chain of $\mathcal{O}_B$ -lattices $\mathcal{L}$ in V giving rise to a $\mathbb{Z}_p$ -model $\mathcal{G}_{\mathcal{L}}$ of G so that $\mathcal{G}_{\mathcal{L}}(\mathbb{Z}_p)$ is a parahoric subgroup of $G(\mathbb{Q}_p)$.

Let $\Phi_F := \text{Hom}_{\mathbb{Q}_p}(F, \bar{\mathbb{Q}}_p)$. We can choose a representative μ of the conjugacy class $\bar{\mu}$ of the form

$$(5.1) \quad \begin{aligned} \mu : \mathbb{G}_{m, \bar{\mathbb{Q}}_p} &\longrightarrow G_{\bar{\mathbb{Q}}_p} = \prod_{\Phi_F} \text{GL}_{n, \bar{\mathbb{Q}}_p} \\ t &\longmapsto (\mu_{\tau}(t))_{\tau \in \Phi_F} \end{aligned}$$

where for each $\tau \in \Phi_F$, $\mu_{\tau}(t) = \text{diag}(t, \dots, t, 1, \dots, 1)$ in which t occurs a_{τ} times. The conjugacy class $\bar{\mu}$ is then uniquely determined by the tuple of integers $(a_{\tau})_{\tau \in \Phi_F}$.

5.1. Globalization.

5.1.1. *CM fields.* Starting from the local objects above, we are going to define some global objects, in particular a global field $\mathbb{F}$ and a group $\mathbb{G}$ over it, among others; we warn the reader that despite the common notation, these objects will not be the same as the ones defined in the Shimura variety context (see section 4.1.1).

Let $\mathbb{F}_0$ be a totally real field such that there is a unique place v of $\mathbb{F}_0$ above p and the v -adic completion $\mathbb{F}_{0,v}$ is isomorphic to the local field F . The existence of such an $\mathbb{F}_0$ is guaranteed by [Far04, Lemme 8.1.2]. Let $\mathbb{K}$ be an imaginary quadratic field in which p is split and let $\mathbb{F} = \mathbb{F}_0\mathbb{K}$ be the CM field with maximal totally real subfield $\mathbb{F}_0$. The complex conjugations on $\mathbb{K}$, $\mathbb{F}$ and $\mathbb{C}$ will all be denoted by the letter “ c ”. Fix a place w of $\mathbb{F}$ above v . Then v splits in $\mathbb{F}$ as $v = ww^c$. We fix field isomorphisms $\mathbb{F}_w \cong \mathbb{F}_{0,v} \cong F$.

Fix a field embedding $\tau_{\mathbb{K}} : \mathbb{K} \hookrightarrow \mathbb{C}$ and let $\Phi = \{\varphi \in \text{Hom}_{\mathbb{Q}}(\mathbb{F}, \mathbb{C}) \mid \varphi_{\mathbb{K}} = \tau_{\mathbb{K}}\}$ be the CM-type of $\mathbb{F}$ determined by $\tau_{\mathbb{K}}$. In particular, we have a decomposition $\text{Hom}_{\mathbb{Q}}(\mathbb{F}, \mathbb{C}) = \Phi \sqcup c\Phi$.

For each place x of $\mathbb{Q}$, we fix a field isomorphism $\iota_x : \mathbb{C} \xrightarrow{\sim} \mathbb{Q}_x$. Then the isomorphism ι_p induces a bijection $\Phi \xrightarrow{\sim} \Phi_F = \text{Hom}_{\mathbb{Q}}(F, \mathbb{Q}_p)$ which permits us to associate to any $\tau \in \Phi$ an integer $a_{\tau} \in \{0, \dots, n\}$ occurring in the definition of the local cocharacter μ in (5.1).

5.1.2. *Division algebras with involution.* We write the invariant of the local division algebra D in the form $\text{inv}(D) = \frac{a}{n}$ for some $a \in \mathbb{Z}$.

Let $\mathbb{D}$ be a central division algebra of dimension n^2 over $\mathbb{F}$ satisfying the following conditions:

- (1) The opposite algebra $\mathbb{D}^{\text{op}}$ is isomorphic to $\mathbb{D} \otimes_{\mathbb{K}, c} \mathbb{K}$ (where both sides are viewed as algebras over $\mathbb{K}$).
- (2) $\mathbb{D}_w \cong B^{\text{op}} = M_m(D)$;
- (3) At any place $x \neq w, w^c$ of $\mathbb{F}$ that is split over $\mathbb{F}_0$, $\mathbb{D}_x$ is either split or a division algebra and there is at least one such place x such that $\mathbb{D}_x$ is a division algebra;
- (4) At any place $x \neq w, w^c$ of $\mathbb{F}$ that is not split over $\mathbb{F}_0$, $\mathbb{D}_x$ is split;
- (5) If n is even, then $a + [\mathbb{F}_0 : \mathbb{Q}]n/2 + \sum_{\tau \in \Phi} a_{\tau}$ is congruent modulo 2 to the number of places $v' \neq v$ of $\mathbb{F}_0$ such that $\mathbb{D}$ is ramified at some place of $\mathbb{F}$ lying above v' .

Let $\mathbb{B} := \mathbb{D}^{\text{op}}$ and let $\mathbb{V} = \mathbb{B}$ be viewed as a left $\mathbb{B} \otimes_{\mathbb{F}} \mathbb{B}^{\text{op}}$ -module via left and right multiplication. That is, for $b_1, b_2, x \in \mathbb{B}$, the action is given by $(b_1 \otimes b_2)(x) = b_1 x b_2$, the product in $\mathbb{B}$. Note that this is equivalent to the set-up in §4.1.2, although the conventions look superficially different.

We choose a *positive* involution $\dagger$ on $\mathbb{B}$ of the second kind (i.e. $\dagger$ restricts to complex conjugation on the center $\mathbb{F}$ and for any $x \in \mathbb{B}^{\times}$ we have $\text{Tr}_{\mathbb{B}/\mathbb{Q}}(xx^{\dagger}) > 0$). This is possible by condition (1) and (4), see the discussion on [HT01, page 51-52]. The condition in (3) that $\mathbb{D}$ is a division algebra over at least one place x of $\mathbb{F}$ that splits over $\mathbb{F}_0$ will be needed in the proof of Theorem 5.2.2: it comes from an assumption in [Lab99, Théorème A.5.2].

5.1.3. *Unitary similitude groups.* Above we consider the pair $(\mathbb{B}, \dagger)$ where $\dagger$ is a positive involution of the second type on $\mathbb{B} = \mathbb{D}^{\text{op}}$. We next need to reverse the procedure used in sections §4.1.1, 4.1.2, and construct from this a suitable involution $*_{\beta}$ on $\mathbb{D}$ (which is not necessarily positive).

For any invertible element $0 \neq \beta \in \mathbb{B}$ such that $\beta^{\dagger} = -\beta$ we can associate the following objects:

- An involution of the second kind $*_{\beta}$ on $\mathbb{B}$ defined by

$$b^{*_{\beta}} := \beta b^{\dagger} \beta^{-1} \quad \forall b \in \mathbb{B}.$$

- A non-degenerate alternating pairing $(\cdot, \cdot)_{\beta}$ on $\mathbb{V}$ defined by

$$(x_1, x_2)_{\beta} := \text{tr}_{\mathbb{B}/\mathbb{Q}}(x_1 \beta x_2^{\dagger}) \quad \forall x_1, x_2 \in \mathbb{V}.$$

Then for all $b_1, b_2 \in \mathbb{B}$ we have $(b_1 x_1 b_2, x_2)_{\beta} = (x_1, b_1^{\dagger} x_2 b_2^{*_{\beta}})_{\beta}$.

- A $\mathbb{Q}$ -algebraic group $\mathbb{G}_{\beta}$ defined as the subgroup of $\text{Aut}_{\mathbb{B}} \mathbb{V}$ preserving the form $(\cdot, \cdot)_{\beta}$ up to a scalar in $\mathbb{Q}$. Since $\text{End}_{\mathbb{B}}(\mathbb{V}) = \mathbb{D}$, for any $\mathbb{Q}$ -algebra R we have

$$\mathbb{G}_{\beta}(R) = \{g \in \mathbb{D} \otimes_{\mathbb{Q}} R \mid xx^{*_{\beta}} \in R^{\times}\}.$$

The map $g \mapsto gg^{*\beta}$ defines a homomorphism of $\mathbb{Q}$ -algebraic groups $c : \mathbb{G}_\beta \rightarrow \mathbb{G}_m$. Let $\mathbb{G}_{\beta,0} \subset \mathbb{G}_\beta$ be its kernel. In other words, for any $\mathbb{Q}$ -algebra R we have

$$\mathbb{G}_{\beta,0}(R) = \{g \in \mathbb{B}^{op} \otimes_{\mathbb{Q}} R \mid xx^{*\beta} = 1\}.$$

Let $\mathbb{G}_\beta^{\text{ad}}$ be the adjoint group of $\mathbb{G}_{\beta,0}$ (and also $\mathbb{G}_\beta$). Sometimes we also view $\mathbb{G}_{\beta,0}$ and $\mathbb{G}_\beta^{\text{ad}}$ as algebraic groups defined over $\mathbb{F}_0$.

Lemma 5.1.1. *There exists $0 \neq \beta \in \mathbb{B}^{\dagger=-1}$ such that*

- *At any finite prime that is non-split in $\mathbb{K}$, $\mathbb{G}_\beta$ and $\mathbb{G}_{\beta,0}$ are quasi-split;*
- *For any infinite place $\tau : \mathbb{F}_0 \hookrightarrow \mathbb{R}$, the pairing $(\cdot, \cdot)_\beta$ on*

$$\mathbb{V} \otimes_{\mathbb{Q}} \mathbb{R} \cong \prod_{\tau \in \text{Hom}_{\mathbb{Q}}(\mathbb{F}_0, \mathbb{R})} \mathbb{V} \otimes_{\mathbb{F}_0, \tau} \mathbb{R}$$

has invariant $(a_\tau, n - a_\tau)$ at the factor corresponding to τ .

Proof. We follow the proof of [HT01, Lemma I.7.1]. First pick any $0 \neq \beta_0 \in \mathbb{B}^{\dagger=-1}$. In this proof, we view the associated groups $\mathbb{G}_{\beta_0,0}$ and $\mathbb{G}_{\beta_0}^{\text{ad}}$ as algebraic groups over $\mathbb{F}_0$. We aim to twist all the objects defined by β_0 by a cohomology class in $H^1(\mathbb{F}/\mathbb{F}_0, \mathbb{G}_{\beta_0}^{\text{ad}})$ so that the conditions in the lemma are satisfied. Concretely, such a cohomology class is represented by an element $0 \neq \alpha \in \mathbb{B}$ with $\alpha^{*\beta_0} = \alpha$ and the twisted objects will be associated to $\beta = \alpha\beta_0$. In particular, $\mathbb{G}_{\alpha\beta_0,0}$ is the inner form of $\mathbb{G}_{\beta_0,0}$ classified by $[\alpha] \in H^1(\mathbb{F}_0, \mathbb{G}_{\beta_0}^{\text{ad}})$.

When n is odd, the natural map of pointed sets

$$H^1(\mathbb{F}_0, \mathbb{G}_{\beta_0}^{\text{ad}}) \rightarrow \bigoplus_x H^1(\mathbb{F}_{0,x}, \mathbb{G}_{\beta_0}^{\text{ad}}),$$

is surjective. Here the direct sum is over all places of $\mathbb{F}_0$. Hence there is no local-global obstruction to finding β with prescribed local conditions.

From now on we assume that n is even. Then there is an exact sequence of pointed sets:

$$(5.2) \quad H^1(\mathbb{F}_0, \mathbb{G}_{\beta_0}^{\text{ad}}) \rightarrow \bigoplus_x H^1(\mathbb{F}_{0,x}, \mathbb{G}_{\beta_0}^{\text{ad}}) \xrightarrow{\text{Obs}} \mathbb{Z}/2\mathbb{Z} \rightarrow 1$$

where Obs is the sum of local obstruction maps $\text{Obs}_x : H^1(\mathbb{F}_{0,x}, \mathbb{G}_{\beta_0}^{\text{ad}}) \rightarrow \mathbb{Z}/2\mathbb{Z}$ whose definition we recall next.

If x is a finite place of $\mathbb{F}_0$ that splits into two different places y, y^c in $\mathbb{F}$, then Obs_x is the composition of the isomorphisms

$$H^1(\mathbb{F}_{0,x}, \mathbb{G}_{\beta_0}^{\text{ad}}) \cong H^1(\mathbb{F}_y, \mathbb{G}_{\beta_0}^{\text{ad}}) \cong H^2(\mathbb{F}_y, \mu_n) \cong \mathbb{Z}/n\mathbb{Z}$$

with the natural projection $\mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z}$.

If x is a finite place of $\mathbb{F}_0$ that does not split in $\mathbb{F}$, then Obs_x is the natural isomorphism $H^1(\mathbb{F}_{x,0}, \mathbb{G}_{\beta_0}^{\text{ad}}) \cong \mathbb{Z}/2\mathbb{Z}$.

If x is an infinite place of $\mathbb{F}_0$, then $H^1(\mathbb{F}_{x,0}, \mathbb{G}_{\beta_0}^{\text{ad}})$ is identified with the set of unordered pairs of non-negative integers (a_x, b_x) with $a_x + b_x = n$. Then the map Obs_x sends such a pair (a_x, b_x) to $a_x - a_{x,0} \bmod 2$, where $(a_{x,0}, n - a_{x,0})$ is the signature of $\mathbb{G}_{\beta_0}$ at x .

For any place x of $\mathbb{F}_0$, let $u_x \in H^1(\mathbb{F}_{x,0}, \mathbb{G}_{\beta_0}^{\text{ad}})$ be the class of the quasi-split inner form of $\mathbb{G}_{\beta_0,0}$. Since u_x comes from the global cohomology class for the quasi-split inner form of $\mathbb{G}_{\beta_0,0}$ over $\mathbb{F}_0$, we have $\sum_x \text{Obs}_x(u_x) = 0$.

Let $\text{Spl}_{\mathbb{F}/\mathbb{F}_0}$ be the set of (finite) places of $\mathbb{F}_0$ that are split in $\mathbb{F}$. Let s be the number of places in $\text{Spl}_{\mathbb{F}/\mathbb{F}_0}$, different from v , above which $\mathbb{D}$ is ramified. By our assumption, $s \geq 1$ and for any place y of $\mathbb{F}$ above such a place x of $\mathbb{F}_0$, $\mathbb{D}_y$ is a division algebra so that $\text{Obs}_x(u_x) = 1$ since n is even. Also we recall that by assumption $\text{Inv}(\mathbb{D}_w) = \frac{a}{n}$ so that $\text{Obs}_v(u_v) = a$. Thus we get

$$\sum_{x \in \text{Spl}_{\mathbb{F}/\mathbb{F}_0}} \text{Obs}_x(u_x) = a + s$$

For any infinite place x , u_x corresponds to the pair $(\frac{n}{2}, \frac{n}{2})$ and hence $\text{Obs}_x(u_x) = a_{x,0} - \frac{n}{2} = a_{x,0} + \frac{n}{2}$. Therefore we get

$$(5.3) \quad a + s + [\mathbb{F}_0 : \mathbb{Q}]n/2 + \sum_{\tau|\infty} a_{\tau,0} + \sum_{x \notin \text{Spl}_{\mathbb{F}/\mathbb{F}_0}, x \nmid \infty} \text{Obs}_x(u_x) = 0.$$

To finish the proof we need to show the existence of a global class $[\alpha] \in H^1(\mathbb{F}_0, \mathbb{G}_{\beta_0}^{\text{ad}})$ whose image in $H^1(\mathbb{F}_{0,x}, \mathbb{G}_{\beta_0}^{\text{ad}})$ equals to

- 0 if $x \in \text{Spl}_{\mathbb{F}/\mathbb{F}_0}$;
- u_x if x is a finite place of $\mathbb{F}_0$ not in $\text{Spl}_{\mathbb{F}/\mathbb{F}_0}$;
- $(a_x, n - a_x)$ if x is an infinite place.

By the exact sequence (5.2) it suffices to show that

$$\sum_{\tau|\infty} (a_{\tau} + a_{\tau,0}) + \sum_{x \notin \text{Spl}_{\mathbb{F}/\mathbb{F}_0}, x \nmid \infty} \text{Obs}_x(u_x) = 0.$$

But this follows from (5.3) and assumption (5) in §5.1.2. $\square$

5.1.4. The PEL data $(\mathbb{B}, \dagger, \mathbb{V}, (\cdot, \cdot), h_0)$. From now on in section §5, we fix an element β as in Lemma 5.1.1 and we often simply write $*$, $\mathbb{G}$, $\mathbb{G}_0$ instead of $*_{\beta}$, $\mathbb{G}_{\beta}$, $\mathbb{G}_{\beta,0}$. We also fix an isomorphism $\mathbb{B}_w \cong M_m(D)^{\text{op}}$ which induces isomorphisms $\mathbb{G}_{0,\mathbb{Q}_p} \cong \text{Res}_{F/\mathbb{Q}_p} G$ and $\mathbb{G}_{\mathbb{Q}_p} \cong \text{Res}_{F/\mathbb{Q}_p} G \times \mathbb{G}_m$, where here $G = \text{GL}_m(D)$ is regarded as a group over $F \cong \mathbb{F}_w$. Let us recall the global setup of §4.1 in this special case.

Let $\mathcal{O}_{\mathbb{B}}$ be a $\mathbb{Z}_{(p)}$ -order in $\mathbb{B}$ such that $\mathcal{O}_{\mathbb{B}} \otimes_{\mathbb{Z}_{(p)}} \mathcal{O}_{\mathbb{F}_w} \cong \mathcal{O}_B$. Then we have an isomorphism

$$\mathcal{O}_{\mathbb{B}} \otimes_{\mathbb{Z}_p} \cong \mathcal{O}_B \times \mathcal{O}_B^{\text{op}} = M_m(\mathcal{O}_D^{\text{op}}) \times M_m(\mathcal{O}_D)$$

where the first factor corresponds to w and the second factor corresponds to w^c .

Recall that in this setting $\mathbb{V} = \mathbb{B}$ is endowed with a symmetric pairing $\langle x, y \rangle = \text{tr}_{\mathbb{B}/\mathbb{Q}}(xy^{\dagger})$. In producing the positive involution $\dagger$ we may assume we started with an involution of the 2nd type $*$ on $\mathbb{D} := \mathbb{B}^{\text{op}}$ and then found, using the argument mentioned in §4.1.1, an element $0 \neq \beta_0 \in \mathbb{B}^{*-1}$ such that $x \mapsto x^{\dagger} = \beta_0 x^* \beta_0^{-1}$ is a positive involution on $\mathbb{B}$. Then we use β_0 to define the $\mathbb{Q}$ -valued alternating form $(\cdot, \cdot)$ on $\mathbb{V} = \mathbb{B}$, by the formula $(x, y) = \text{tr}_{\mathbb{B}/\mathbb{Q}}(x\beta_0 y^*)$. From β_0 , we then construct β as in Lemma 5.1.1.

From the lattice chain $\mathcal{L}$ in $V \cong \mathbb{V}_{\mathbb{F}_w}$, we obtain a $(\cdot, \cdot)$ -self-dual chain $\mathcal{L}_i^+$ of $\mathcal{O}_{\mathbb{B}}$ -lattices in $\mathbb{V}_{\mathbb{Q}_p} = \mathbb{V}_{F_w} \oplus \mathbb{V}_{F_w^c} \cong V \oplus V^*$ following the method from §4.1.3.

5.1.5. The Shimura data. We now consider a Shimura data $(\mathbb{G}, X)$, where as above $\mathbb{G} = \mathbb{G}_{\beta} = \text{GU}(\mathbb{D}, *)_{\beta}$ and where X comes from a choice of $*$ -homomorphism $h_0 : \text{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m) \rightarrow \mathbb{G}_{\mathbb{R}}$, which may be constructed as in [HT01, p.91-92] using our choice of CM-type Φ we fixed above. The PEL data for this Shimura variety is given by the tuple $(\mathbb{B}, \dagger, \mathbb{V} = \mathbb{B}, (\cdot, \cdot), h_0)$. Then this choice of Shimura data gives rise to a conjugacy class of cocharacters

$$\begin{aligned} \mu_{\mathbb{C}} : \mathbb{C}^{\times} &\longrightarrow \mathbb{G}(\mathbb{C}) \cong \prod_{\Phi} \text{GL}_n(\mathbb{C}) \times \mathbb{C}^{\times} \\ z &\longmapsto ((\mu_{\mathbb{C},\tau}(z))_{\tau \in \Phi}, z). \end{aligned}$$

We can choose a representative $\mu_{\mathbb{C}}$ such that for any $\tau \in \Phi$, $\mu_{\mathbb{C},\tau}(z) = \text{diag}(z, \dots, z, 1, \dots, 1)$ where z occurs a_{τ} times (see Lemma 5.1.1). The global reflex field $\mathbb{E}$ is the field of definition of the conjugacy class $\mu_{\mathbb{C}}$.

The isomorphism $\iota_p : \mathbb{C} \cong \bar{\mathbb{Q}}_p$ determines a finite place $\mathfrak{p}$ of $\mathbb{E}$ above p . The completion $\mathbb{E}_{\mathfrak{p}}$ is isomorphic to the local reflex field E and we fix a field isomorphism $\mathbb{E}_{\mathfrak{p}} \cong E$. Then the isomorphism ι_p takes the conjugacy class $(\mu_{\mathbb{C},\tau}(z))_{\tau \in \Phi}$ of cocharacters of $\mathbb{G}_0(\mathbb{C})$ to the local conjugacy class $\bar{\mu}$ of $\mathbb{G}_0(\bar{\mathbb{Q}}_p)$.

5.2. Base change of automorphic representations.

5.2.1. *The Langlands dual group.* Let $\mathbb{G}_{\mathbb{K}}$ be the base change of $\mathbb{G}$ to $\mathbb{K}$ and denote $\tilde{\mathbb{G}} := \text{Res}_{\mathbb{K}/\mathbb{Q}} \mathbb{G}_{\mathbb{K}}$. The complex conjugation $c \in \text{Gal}(\mathbb{K}/\mathbb{Q})$ induces an automorphism θ of $\tilde{\mathbb{G}}$ defined over $\mathbb{Q}$ and $\mathbb{G}$ is the closed subgroup of $\tilde{\mathbb{G}}$ consisting of θ -fixed elements. There is a natural isomorphism

$$(5.4) \quad \tilde{\mathbb{G}} \cong \mathbb{B}^{\text{op}, \times} \times \text{Res}_{\mathbb{K}/\mathbb{Q}} \mathbb{G}_m$$

whose composition with the embedding $\mathbb{G} \hookrightarrow \tilde{\mathbb{G}}$ is given on points by $g \mapsto (g, gg^*)$ where the second factor lies in $\mathbb{G}_m \subset \text{Res}_{\mathbb{K}/\mathbb{Q}} \mathbb{G}_m$. Here and throughout this subsection, we abbreviate by writing g^* in place of $g^{*\beta}$. Using this isomorphism, the automorphism θ of $\tilde{\mathbb{G}}$ can be described by $\theta(g, \lambda) := (\lambda^c g^{-*}, \lambda^c)$ where c denotes the automorphism of $\text{Res}_{\mathbb{K}/\mathbb{Q}} \mathbb{G}_m$ induced by complex conjugation in $\text{Gal}(\mathbb{K}/\mathbb{Q})$.

The Langlands dual group ${}^L\mathbb{G} = \mathbb{G}^\vee \rtimes \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ is described explicitly as follows. Using the CM-type Φ , its identity component can be described by $\mathbb{G}^\vee = \prod_{\Phi} \text{GL}_n(\mathbb{C}) \times \mathbb{C}^\times$ and the action of $\sigma \in \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on $\mathbb{G}^\vee$ is defined by $\sigma((g_\varphi)_{\varphi \in \Phi}, z) = ((h_\varphi)_{\varphi \in \Phi}, z')$ where

$$h_\varphi = \begin{cases} g_{\sigma^{-1}\varphi} & \text{if } \sigma^{-1}\varphi \in \Phi \\ w_n {}^t g_{\sigma^{-1}\varphi}^{-1} w_n^{-1} & \text{if } \sigma^{-1}\varphi \notin \Phi \end{cases}$$

(here w_n is the $n \times n$ matrix with entries $(w_n)_{ij} = (-1)^i \delta_{n+1-i, j}$) and

$$z' = z \prod_{\varphi \in \Phi, \sigma^{-1}\varphi \notin \Phi} \det g_\varphi.$$

Similarly, the Langlands dual group of $\tilde{\mathbb{G}}$ is ${}^L\tilde{\mathbb{G}} = \tilde{\mathbb{G}}^\vee \rtimes \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ where

$$\tilde{\mathbb{G}}^\vee = \prod_{\text{Hom}_{\mathbb{Q}}(\mathbb{F}, \mathbb{C})} \text{GL}_n(\mathbb{C}) \times \mathbb{C}^\times \times \mathbb{C}^\times$$

and the action of $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ factors through the quotient $\text{Gal}(\mathbb{K}/\mathbb{Q})$ with the complex conjugation acting by

$$c((g_\varphi)_\varphi, z_1, z_2) = ((g_{c\varphi})_\varphi, z_2, z_1)$$

Define an L -embedding $\eta_{BC} : {}^L\mathbb{G} \rightarrow {}^L\tilde{\mathbb{G}}$ by

$$((g_\varphi)_{\varphi \in \Phi}, z) \rtimes \sigma \mapsto ((h_\varphi)_{\varphi \in \Phi \sqcup c\Phi}, z, z \prod_{\varphi \in \Phi} \det(g_\varphi)) \rtimes \sigma$$

where

$$h_\varphi = \begin{cases} g_\varphi & \text{if } \varphi \in \Phi \\ w_n {}^t g_{c\varphi}^{-1} w_n^{-1} & \text{if } \varphi \notin \Phi. \end{cases}$$

Our goal is to define base change of automorphic representations of $\mathbb{G}$ to $\tilde{\mathbb{G}}$ for the L -embedding η_{BC} .

5.2.2. *Local base change at split primes.* Let x be a prime of $\mathbb{Q}$ that splits in $\mathbb{K}$ as $x = yy^c$. The canonical isomorphism $\mathbb{Q}_x \cong \mathbb{K}_y$ allows us to view $\mathbb{Q}_x$ as a $\mathbb{K}$ -algebra and from (5.4) we obtain an isomorphism

$$(5.5) \quad \mathbb{G}(\mathbb{Q}_x) \cong (\mathbb{B}_y^{\text{op}})^\times \times \mathbb{Q}_x^\times.$$

Let π be an irreducible admissible representation of $\mathbb{G}(\mathbb{Q}_x)$. Then we can decompose it as $\pi \cong \pi_y \otimes \chi_x$ where π_y is an irreducible admissible representation of $(\mathbb{B}_y^{\text{op}})^\times$ and χ_x is a character of $\mathbb{Q}_x^\times$. Let ω_π be the central character of π and decompose its restriction to $\mathbb{K}_x^\times \cong \mathbb{K}_y^\times \times \mathbb{K}_{y^c}^\times$ as $\omega_{\pi, y} \otimes \omega_{\pi, y^c}$. Similarly, let ω_{π_y} be the central character of π_y .

Lemma 5.2.1. *With notations as above, we have $\chi_x = \omega_{\pi, y^c}$ and $\omega_{\pi, y} = \omega_{\pi_y} \omega_{\pi, y^c}$, all viewed as characters of $\mathbb{Q}_x^\times$ via the isomorphisms $\mathbb{Q}_x \cong \mathbb{K}_y \cong \mathbb{K}_{y^c}$. In particular, we have $\pi \cong \pi_y \otimes \omega_{\pi, y^c}$.*

Proof. Any $(a_1, a_2) \in \mathbb{K}_x^\times \cong \mathbb{K}_y^\times \times \mathbb{K}_{y^c}^\times$ embedded in the center of $\mathbb{G}(\mathbb{Q}_x)$ acts on the representation π by the scalar $\pi(a_1, a_2) = \omega_{\pi, y}(a_1)\omega_{\pi, y^c}(a_2)$. On the other hand, under the isomorphism (5.5), (a_1, a_2) is sent to $(a_1, a_1 a_2) \in (\mathbb{B}_y^{op})^\times \times \mathbb{K}_x^\times$ and hence from the decomposition $\pi \cong \pi_y \otimes \chi_x$ we get $\pi(a_1, a_2) = \omega_{\pi_y}(a_1)\chi_x(a_1 a_2)$. Consequently, after the identifications $\mathbb{Q}_x \cong \mathbb{K}_y \cong \mathbb{K}_{y^c}$ we obtain

$$\omega_{\pi, y}(a_1)\omega_{\pi, y^c}(a_2) = \omega_{\pi_y}(a_1)\chi_x(a_1 a_2), \quad \forall a_1, a_2 \in \mathbb{Q}_x$$

Let $a_1 = 1$ we get $\chi_x = \omega_{\pi, y^c}$ and let $a_2 = 1$ we get $\omega_{\pi, y} = \omega_{\pi_y}\omega_{\pi, y^c}$. $\square$

We define the local base change of π to be the representation

$$BC(\pi) = \pi_y \otimes \pi_y^\# \otimes (\omega_{\pi, y^c} \circ c) \otimes (\omega_{\pi, y} \circ c)$$

of

$$\tilde{\mathbb{G}}(\mathbb{Q}_x) = \mathbb{G}(\mathbb{K}_x) \cong (\mathbb{B}_y^{op})^\times \times (\mathbb{B}_{y^c}^{op})^\times \times \mathbb{K}_y^\times \times \mathbb{K}_{y^c}^\times$$

where $\pi_y^\#(g) := \pi_y(g^{-*})$.

Let $w_1, \dots, w_s$ be the primes of $\mathbb{F}$ above y . Then we have a decomposition $\mathbb{B}_y = \prod_{i=1}^s \mathbb{B}_{w_i}$ and $\pi_y = \otimes_{i=1}^s \pi_{w_i}$ where π_{w_i} is a representation of $(\mathbb{B}_{w_i}^{op})^\times$. Consequently corresponding to the decomposition

$$\tilde{\mathbb{G}}(\mathbb{Q}_x) \cong \prod_{i=1}^s (\mathbb{B}_{w_i}^{op})^\times \times \prod_{i=1}^s (\mathbb{B}_{w_i^c}^{op})^\times \times \mathbb{K}_y^\times \times \mathbb{K}_{y^c}^\times$$

we can write

$$BC(\pi) = \otimes_{i=1}^s BC(\pi)_{w_i} \otimes \otimes_{i=1}^s BC(\pi)_{w_i^c} \otimes \chi_{BC(\pi), y} \otimes \chi_{BC(\pi), y^c}$$

where $BC(\pi)_{w_i} := \pi_{w_i}$, $BC(\pi)_{w_i^c} := \pi_{w_i}^\#$, $\chi_{BC(\pi), y} := \omega_{\pi, y^c} \circ c$ and $\chi_{BC(\pi), y^c} := \omega_{\pi, y} \circ c$.

5.2.3. Unramified local base change at inert primes. Now let x be a prime of $\mathbb{Q}$ such that

- x is inert in $\mathbb{K}$ and unramified in $\mathbb{F}$;
- $(\mathbb{B}_x^{op}, *) \cong (M_n(\mathbb{F}_x), \dagger)$ where $g^\dagger := w_n^t g^c w_n^{-1}$.

The isomorphism $\iota_x : \mathbb{C} \xrightarrow{\sim} \bar{\mathbb{Q}}_x$ induces an embedding from the Weil group $W_{\mathbb{Q}_x}$ into $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ (here $\bar{\mathbb{Q}}$ is the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$). We use ι_x to identify the CM-type Φ with the subset of $\text{Hom}_{\mathbb{Q}_x}(\mathbb{F}_x, \bar{\mathbb{Q}}_x)$ consisting of homomorphisms φ such that $\varphi|_{\mathbb{K}} = \iota_x \circ \tau_{\mathbb{K}}$.

The group $\mathbb{G}(\mathbb{Q}_x)$ is unramified and contains a standard maximal torus T_x and Borel subgroup B_x defined over $\mathbb{Q}_x$, consisting of diagonal and upper triangular matrices. The torus T_x can be described as

$$T_x = \{(t_1, \dots, t_n; t_0) \in (\mathbb{F}_x^\times)^n \times \mathbb{Q}_x^\times \mid t_0 = t_i t_{n+1-i}^c, \forall i = 1, \dots, n\}.$$

Similarly, $\tilde{\mathbb{G}}_x \cong \text{GL}_n(\mathbb{F}_x) \times \mathbb{K}_x^\times$ contains the diagonal maximal torus $\tilde{T}_x \cong (\mathbb{F}_x^\times)^n \times \mathbb{K}_x^\times$ and standard Borel subgroup $\tilde{B}_x = B_n(\mathbb{F}_x) \times \mathbb{K}_x^\times$ where $B_n(\mathbb{F}_x)$ is the group of upper triangular matrices in $\text{GL}_n(\mathbb{F}_x)$. The automorphism θ on $\tilde{\mathbb{G}}$ induces automorphisms of $\tilde{T}_x$ and $\tilde{B}_x$ so that $T_x = \tilde{T}_x^\theta$ and $B_x = \tilde{B}_x^\theta$ are identified as the fixed point subgroups.

For any smooth irreducible unramified representation π of $\mathbb{G}(\mathbb{Q}_x)$, there is an unramified character χ of T_x such that π is the unique unramified subquotient of the normalized parabolic induction $i_{B_x}^{\mathbb{G}(\mathbb{Q}_x)} \chi$. The character χ corresponds to an element $\alpha = ((\alpha_{1, \varphi}, \dots, \alpha_{n, \varphi})_{\varphi \in \Phi}; \alpha_0) \in \mathbb{T}^\vee = \prod_{\varphi \in \Phi} (\mathbb{C}^\times)^n \times \mathbb{C}^\times$. More precisely, if $t = (t_1, \dots, t_n; t_0) \in T_x$, then

$$\chi(t) = \alpha_0^{\text{val}_x(t_0)} \prod_{\varphi \in \Phi} \prod_{i=1}^n (\alpha_{i, \varphi})^{\text{val}_x(\iota_x \circ \varphi(t_i))}$$

where val_x denotes the x -adic valuation on $\bar{\mathbb{Q}}_x$.

The Langlands parameter for π is the conjugacy class of unramified L -homomorphism $\varphi_\pi : W_{\mathbb{Q}_x} \rightarrow \mathbb{G}^\vee \rtimes W_{\mathbb{Q}_x}$ such that $\varphi_\pi(\text{Frob}_x) = \alpha \rtimes \text{Frob}_x$ for any geometric Frobenius $\text{Frob}_x \in W_{\mathbb{Q}_x}$. The base change $BC(\pi)$ is the unramified representation of $\tilde{\mathbb{G}}_x \cong \text{GL}_n(\mathbb{F}_x) \times \mathbb{K}_x^\times$ whose Langlands parameter is $\eta_{BC} \circ \varphi$. In other words, $BC(\pi)$ is the unique irreducible unramified subquotient of

the normalized induction $i_{\tilde{B}^x}^{\tilde{G}_x} \tilde{\chi}$ where $\tilde{\chi}$ is the character of $\tilde{T}_x$ corresponding to $\eta_{BC}(\alpha) \in \mathbb{T}^\vee$. More explicitly, for any $(t_1, \dots, t_n; t_0) \in (\mathbb{F}_x^\times)^n \times \mathbb{K}_x^\times$, we have

$$\tilde{\chi}(t_1, \dots, t_n; t_0) = \chi(t_0^c t_1 / t_n^c, \dots, t_0^c t_n / t_1^c; t_0 t_0^c).$$

Let $w_1, \dots, w_s$ be the primes of $\mathbb{F}$ above x . For each $i = 1, \dots, s$, let $\Phi_{w_i} := \{\varphi \in \Phi, \varphi|_{\mathbb{F}_{w_i}} \neq 0\}$ so that $\Phi = \sqcup_{i=1}^s \Phi_{w_i}$. Corresponding to the decomposition $\mathbb{F}_x = \prod_{i=1}^s \mathbb{F}_{w_i}$ we have an isomorphism

$$\tilde{G}_x \cong \prod_{i=1}^s \mathrm{GL}_n(\mathbb{F}_{w_i}) \times \mathbb{K}_x^\times$$

which induces a decomposition

$$BC(\pi) = \otimes_{i=1}^s BC(\pi)_{w_i} \otimes \chi_{BC(\pi)}$$

where

- For each $i = 1, \dots, s$, $BC(\pi)_{w_i}$ is an unramified representation of $\mathrm{GL}_n(\mathbb{F}_{w_i})$ with Satake parameter

$$\left(\prod_{\varphi \in \Phi_{w_i}} \alpha_{1,\varphi} \alpha_{n,\varphi}^{-1}, \dots, \prod_{\varphi \in \Phi_{w_i}} \alpha_{n,\varphi} \alpha_{1,\varphi}^{-1} \right),$$

where the j -th coordinate is $\prod_{\varphi \in \Phi_{w_i}} \alpha_{j,\varphi} \alpha_{n+1-j,\varphi}^{-1}$ for any $1 \leq j \leq n$;

- $\chi_{BC(\pi)}$ is the unramified character of $\mathbb{K}_x^\times$ sending a uniformizer to $\alpha_0^2 \prod_{i=1}^n \prod_{\varphi \in \Phi} \alpha_{i,\varphi}$.

5.2.4. Global base change. Let ξ be an irreducible algebraic representation of $\mathbb{G}(\mathbb{C})$ on a finite dimensional $\mathbb{C}$ -vector space. Recall that we have a decomposition

$$\tilde{G}(\mathbb{C}) = \mathbb{G}(\mathbb{K} \otimes_{\mathbb{Q}} \mathbb{C}) \cong \mathbb{G}(\mathbb{C}) \times \mathbb{G}(\mathbb{C})$$

where the first factor corresponds to the embedding $\tau_{\mathbb{K}}$ and the second factor corresponds to $\tau_{\mathbb{K}}^c$. With this identification, we define $\xi_{\mathbb{K}}$ to be the representation $\xi \otimes \xi$ of $\tilde{G}(\mathbb{C})$.

Theorem 5.2.2. *Let π be an irreducible automorphic representation of $\mathbb{G}(\mathbb{A})$ whose archimedean component π_∞ is cohomological for ξ . Let ω_π denote the central character of π . Then there is a unique irreducible automorphic representation $BC(\pi) = (\Pi, \omega)$ of $\tilde{G}(\mathbb{A}) = (\mathbb{B}^{op} \otimes_{\mathbb{Q}} \mathbb{A})^\times \times \mathbb{A}_{\mathbb{K}}^\times$ such that*

- (1) $\omega = \omega_\pi^c|_{\mathbb{A}_{\mathbb{K}}^\times}$ and $\omega_\infty^c = (\xi|_{\mathbb{K}_\infty^\times})^{-1}$
- (2) For any place x of $\mathbb{Q}$ that splits in $\mathbb{K}$, $BC(\pi)_x = BC(\pi_x)$
- (3) For all but finitely many places x of $\mathbb{Q}$ that are inert in $\mathbb{K}$, we have $BC(\pi)_x = BC(\pi_x)$;
- (4) Π_∞ is cohomological for $\xi_{\mathbb{K}}$;
- (5) If ω_Π is the central character of Π , then $\omega_\Pi|_{\mathbb{A}_{\mathbb{K}}^\times} = \omega^c \omega^{-1}$;
- (6) $\Pi^\# \cong \Pi$ where $\Pi^\#(g) := \Pi(g^{-*})$.

Proof. This is basically Theorem VI.2.1 of [HT01], which improves upon Theorem A.5.2 of [Lab99]. We briefly review the argument here.

First one reduces to base change for the unitary group $\mathbb{G}_1$ instead of the unitary similitude group $\mathbb{G}$. More precisely, one shows that there exists an automorphic representation π' of $\mathbb{G}_1(\mathbb{A})$ such that

- For any place x of $\mathbb{Q}$ that splits in $\mathbb{K}$, $\pi'_x \cong \pi_x|_{\mathbb{G}_1(\mathbb{Q}_x)}$
- For all but finitely many places x of $\mathbb{Q}$ that are inert in $\mathbb{K}$ and unramified in $\mathbb{F}$, π'_x is the unique unramified subquotient of $\pi_x|_{\mathbb{G}_1(\mathbb{Q}_x)}$.

Then one applies Theorem A.5.2 of [Lab99] (which is slightly generalized in the proof of Theorem VI.2.1 of [HT01]) to π' to obtain Π . The crucial ingredient in this result is a comparison of a simple twisted trace formula for $\tilde{\mathbb{G}}_1 := \mathrm{Res}_{\mathbb{K}/\mathbb{Q}} \mathbb{G}_{1,\mathbb{K}} = \mathbb{B}^{op,\times}$ and a simple trace formula for $\mathbb{G}_1$. The comparison goes by equating both trace formulas with the stable trace formula for the quasi-split inner form $\mathbb{G}_1^*$ with suitable choice of test functions. More precisely, we choose $\phi = \otimes_v \phi_v \in C_c^\infty(\tilde{\mathbb{G}}_1)$ and $f = \otimes_v f_v \in C_c^\infty(\mathbb{G}_1)$ satisfying:

- For each place v , ϕ_v and f_v have matching stable (twisted) orbital integrals;
- ϕ_∞ is the Lefschetz function associated to $\tilde{\xi}$ and the automorphism θ on $\tilde{\mathbb{G}}_{1,\infty}$ and f_∞ ;
- f_∞ is, up to a positive constant multiple, the Euler-Poincaré function associated to ξ .

It follows from Theorem A.1.1 and Corollaire 1.2 of [Lab99] that ϕ_∞ and f_∞ have matching stable (twisted) orbital integrals and the unstable twisted orbital integrals of ϕ_∞ vanishes. We remark that in [Lab99] it is assumed that ξ is trivial and this assumption is removed in [CL11] (which also corrects some sign errors in [Lab99]). The vanishing of unstable twisted orbital integrals of ϕ_∞ implies that the twisted trace formula for $\tilde{\mathbb{G}}_1$ equals a stable trace formula for $\mathbb{G}_1^*$. From the assumption that $\mathbb{B}_x$ is a division algebra for some place x of $\mathbb{F}$ that splits over $\mathbb{F}_0$, it follows by an observation of Kottwitz that the trace formula of $\mathbb{G}_1$ equals the stable trace formula for $\mathbb{G}_1^*$. Combining these two comparisons one obtains the sought-after comparison of simple (twisted) trace formulas of $\tilde{\mathbb{G}}_1$ and $\mathbb{G}_1$ as in Theorem A.3.1 of [Lab99].

Using this comparison one proceeds as in the proof of Theorem VI.2.1 of [HT01] to get the automorphic representation Π satisfying the desired properties. We remark that property (2) is not explicitly stated in [Lab99] but follows from the generalization of its arguments in [HT01]. $\square$

5.3. Description of Cohomology. The following result establishes the existence of Galois representations (satisfying suitable local-global compatibility conditions) associated to cohomological automorphic representations of $\mathbb{G}$.

Theorem 5.3.1. *Let π be an irreducible automorphic representation of $\mathbb{G}(\mathbb{A})$ such that π_∞ is cohomological for ξ . There exists an n -dimensional ℓ -adic Galois representation $\sigma_\ell(\pi)$ of $\text{Gal}(\mathbb{F}/\mathbb{F})$ such that*

- (1) *For any place w of $\mathbb{F}$ that splits over $\mathbb{F}_0$, the restriction of $\sigma_\ell(\pi)$ to the Weil group of $\mathbb{F}_w$ is isomorphic to the ℓ -adic representation associated to the irreducible smooth representation π_w of $\mathbb{B}_w^{\text{op},\times}$ under the local Langlands correspondence.*
- (2) *For all but finitely many places w of $\mathbb{F}$ such that w is inert over $\mathbb{F}_0$ and unramified over $\mathbb{Q}$ and π is unramified at $x = w|_{\mathbb{Q}}$, the restriction of $\sigma_\ell(\pi)$ to the Weil group of $\mathbb{F}_w$ is the unramified representation corresponding to the irreducible smooth representation $BC(\pi_x)_w$ of $\text{GL}_n(\mathbb{F}_w)$.*

Proof. Consider the base change $BC(\pi) = (\Pi, \omega)$ from Theorem 5.2.2. Let $JL(\Pi)$ be the automorphic representation of $\text{GL}_n(\mathbb{A}_{\mathbb{F}})$ associated to Π by the global Jacquet-Langlands correspondence established in [Bad08] (note that Π is automatically cuspidal). Then $JL(\Pi)$ is conjugate self-dual and the archimedean component $JL(\Pi)_\infty$ is regular algebraic. Moreover, by our choice of $\mathbb{B}$, there is a place x of $\mathbb{F}$ that splits over $\mathbb{F}_0$ such that $\mathbb{B}_x$ is a division algebra, hence $JL(\Pi)_x$ is square-integrable. Thus we can apply Theorem VII.1.9 of [HT01] to $JL(\Pi)$. In the notation of *loc.cit.* we obtain an ℓ -adic Galois representation $\sigma_\ell(\pi) := R_\ell(JL(\Pi))$ that satisfies the conditions. $\square$

We will apply the above to get an understanding of the cohomology of the Shimura variety $Sh(\mathbb{G}_\beta, X)$ described in §5.1.4. This is another simple Shimura variety of the type considered by Kottwitz, but unlike the original general set-up we started with, the nature of the group $\mathbb{G} = \mathbb{G}_\beta$ in this special case allows us to exploit the Galois representations mentioned above.

Let $H_{\beta,\xi}^* := H^*(Sh(\mathbb{G}_\beta, X)_{\overline{\mathbb{Q}}}, \mathcal{L}_\xi)$. We have a decomposition

$$H_{\beta,\xi}^* = \bigoplus_{\pi_f} \pi_f \otimes \sigma_{\beta,\xi}(\pi_f)$$

where $\sigma_{\beta,\xi}(\pi_f)$ is a virtual representation of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{E})$.

Corollary 5.3.2. *In the setting of §5.1, there is an isomorphism of virtual $W_E = W_{\mathbb{E}_p}$ representations*

$$\sigma_{\beta,\xi}(\pi_f)|_{W_{\mathbb{E}_p}} \cong a(\pi_f)(r_{-\mu} \circ \varphi_{\pi_p}|_{W_{\mathbb{E}_p}}) \cdot | \cdot |^{-\dim \text{Sh}(\mathbb{G}_\beta, X)/2}.$$

Proof. Let $\tilde{\mathbb{E}}$ be the composite of $\mathbb{E}$ and the imaginary quadratic field $\mathbb{K}$ in $\mathbb{C}$ where we use the embedding $\tau_{\mathbb{K}}$ fixed in § 5.1.1 to view $\mathbb{K}$ as a subfield of $\mathbb{C}$. Since p is split in $\mathbb{K}$, there is a place $\tilde{p}$

of $\tilde{\mathbb{E}}$ above $\mathfrak{p}$ and an isomorphism $\tilde{\mathbb{E}}_{\mathfrak{p}} \cong \mathbb{E}_{\mathfrak{p}}$. Using Theorem 5.3.1 and the arguments in the proof of [Far04, Th  orem   A.7.2], one can construct an ℓ -adic representation of $\text{Gal}(\overline{\mathbb{Q}}/\tilde{\mathbb{E}})$ that agrees with $\sigma_{\beta,\xi}(\pi_f)$ at all unramified places, then we can conclude by the Chebotarev density theorem. See also [SS13, proof of Theorem 8.1].

Let us explain the arguments in more detail. Let $\sigma_{\ell}(\pi)$ be the n -dimensional representation of $\text{Gal}(\overline{\mathbb{F}}/\mathbb{F})$ from Theorem 5.3.1. Under the Shapiro isomorphism, $\sigma_{\ell}(\pi)$ corresponds to a (conjugacy class of) homomorphism(s):

$$\varphi_{\ell}(\pi) : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{K}) \rightarrow \text{GL}_n(\overline{\mathbb{Q}}_{\ell})^{\text{Hom}_{\mathbb{K}}(\mathbb{F}, \overline{\mathbb{Q}})} \rtimes \text{Gal}(\overline{\mathbb{Q}}/\mathbb{K}) = \text{GL}_n(\overline{\mathbb{Q}}_{\ell})^{\Phi} \rtimes \text{Gal}(\overline{\mathbb{Q}}/\mathbb{K}).$$

Let ω_{π} be the central character of π and let $\omega = \omega_{\pi}|_{\mathbb{A}_{\mathbb{K}}^{\times}}$ be as in Theorem 5.2.2. Then ω is an algebraic Hecke character and let $\sigma_{\ell}(\omega) : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{K}) \rightarrow \overline{\mathbb{Q}}_{\ell}^{\times}$ be the associated (one-dimensional) ℓ -adic Galois representation. Combining the restriction of $\varphi_{\ell}(\pi)$ and $\sigma_{\ell}(\omega)$ to $\text{Gal}(\overline{\mathbb{Q}}/\tilde{\mathbb{E}})$ we get a homomorphism

$$\tilde{\varphi}_{\ell}(\pi) : \text{Gal}(\overline{\mathbb{Q}}/\tilde{\mathbb{E}}) \rightarrow {}^L G_{\ell} = (\text{GL}_n(\overline{\mathbb{Q}}_{\ell})^{\Phi} \times \overline{\mathbb{Q}}_{\ell}^{\times}) \rtimes \text{Gal}(\overline{\mathbb{Q}}/\tilde{\mathbb{E}}).$$

By construction, for any finite prime $\tilde{\mathfrak{q}}$ of $\tilde{\mathbb{E}}$ with residue characteristic q , we have

$$\tilde{\varphi}_{\ell}(\pi)|_{\tilde{\mathbb{E}}_{\tilde{\mathfrak{q}}}} \cong \varphi_{\pi_q}|_{\tilde{\mathbb{E}}_{\tilde{\mathfrak{q}}}}$$

Consider the virtual representation $R(\pi)$ of $\text{Gal}(\overline{\mathbb{Q}}/\tilde{\mathbb{E}})$ defined by

$$R(\pi) := a(\pi_f)(r_{-\mu} \circ \tilde{\varphi}_{\ell}(\pi))| \cdot |^{-\dim \text{Sh}(\mathbb{G}_{\beta}, X)/2}.$$

For any prime $\tilde{\mathfrak{q}}$ of $\tilde{\mathbb{E}}$ with residue characteristic q such that π_q is unramified and the Shimura variety has good reduction at q , we have $\sigma_{\beta,\xi}(\pi_f)|_{\tilde{\mathbb{E}}_{\tilde{\mathfrak{q}}}} \cong R(\pi)|_{\tilde{\mathbb{E}}_{\tilde{\mathfrak{q}}}}$ by Kottwitz [Kot92a]. By the Chebotarev density theorem we get $\sigma_{\beta,\xi}(\pi_f)|_{\text{Gal}(\overline{\mathbb{Q}}/\tilde{\mathbb{E}})} \cong R(\pi)$ (at least when restricted to all local Weil groups and after semisimplifying the Weil group actions). In particular restricting to the Weil group of $\tilde{\mathbb{E}}_{\mathfrak{p}} \cong \mathbb{E}_{\mathfrak{p}}$ we get the desired isomorphism. $\square$

Recall from Definition 3.4.1 the element $z_{\tau,-\mu}$ in the stable Bernstein center of $\mathbb{G}(\mathbb{Q}_p)$.

Corollary 5.3.3. *Let $r \geq 1$ be an integer and $\tau \in \text{Frob}_E^r I_E \subset W_E$, where $E \cong \mathbb{E}_{\mathfrak{p}}$ is the local reflex field. For any $h \in C_c^{\infty}(\mathcal{G}_{\mathcal{L}}(\mathbb{Z}_p))$ and $f^p \in C_c^{\infty}(\mathbb{G}(\mathbb{A}_f^p))$, we have*

$$\text{tr}(\tau \times h f^p | H_{\beta,\xi}^*) = \sum_{\pi} m(\pi) \text{Tr}(f_{\tau,h} f^p f_{\xi,\infty}^{\mathbb{G}} | \pi)$$

where $f_{\tau,h} := z_{\tau,-\mu} * h$ and the sum ranges over automorphic representations of $\mathbb{G}$.

Proof. By Corollary 5.3.2, the left hand side is equal to

$$\sum_{\pi_f} a(\pi_f) \text{Tr}(f_{\tau,h} f^p | \pi_f).$$

This is equal to the right hand side by the definition of $a(\pi_f)$, see (4.11). $\square$

5.4. Fixed points via generalized Kottwitz triples. We would like to proceed from Corollary 4.4.2 and find an expression of the right hand side in terms of a trace formula, without knowing the vanishing property of the test function $\phi_{\tau,h}$. For this purpose, we will associate to each fixed point of the Frobenius-Hecke correspondence a substitute of the Kottwitz triple defined using an auxiliary group $\mathbb{G}'$ that is constructed in a similar way to $\mathbb{G}$.

5.4.1. *An auxiliary group.* Let $\mathbb{B}'$ be the division algebra over $\mathbb{F}$ that is split at w, w^c and isomorphic to $\mathbb{B} = \mathbb{D}^{\text{op}}$ at all other finite places of $\mathbb{F}$. Choose a *positive* involution $\dagger'$ of the second kind on $\mathbb{G}'$. Analogously to the construction of $\mathbb{G}$ in Lemma 5.1.1, we can find an element $0 \neq \beta' \in \mathbb{B}'$ that, together with the positive involution $\dagger'$, gives rise to an involution $*'$, a $\dagger'$ -Hermitian pairing on $\mathbb{V}' = \mathbb{B}'$, and a group $\mathbb{G}'$ such that

- $\mathbb{G}'_{\mathbb{Q}_p} \cong \text{Res}_{F/\mathbb{Q}_p} \text{GL}_n \times \mathbb{G}_m$ is the quasi-split inner form of $\mathbb{G}_{\mathbb{Q}_p}$;
- For any finite prime $\ell \neq p$, we have $\mathbb{G}'_{\mathbb{Q}_\ell} \cong \mathbb{G}_{\mathbb{Q}_\ell}$;

Note that we do not put any condition at the archimedean places.

5.4.2. *Generalized Kottwitz triples.*

Definition 5.4.1. Let $j \in \mathbb{Z}_{\geq 1}$ and set $r := j[\kappa_{\mathbb{E}_p} : \mathbb{F}_p]$. A degree j *generalized Kottwitz triple* $(\gamma'_0; \gamma, \delta)$ consists of

- A semisimple stable conjugacy class $\gamma'_0 \in \mathbb{G}'(\mathbb{Q})$ that is elliptic in $\mathbb{G}'(\mathbb{R})$,
- a conjugacy class $\gamma \in \mathbb{G}'(\mathbb{A}_f^p) = \mathbb{G}(\mathbb{A}_f^p)$ that is stably conjugate to γ'_0 ,
- a σ -conjugacy class $\delta \in \mathbb{G}(\mathbb{Q}_{p^r})$ whose naive norm $N_r \delta = \delta \sigma(\delta) \cdots \sigma^{r-1}(\delta)$ is stably conjugate to γ'_0 , such that $\kappa_{\mathbb{G}_{\mathbb{Q}_p}}(p\delta) = -\mu^\#$ in $X^*(Z(\hat{\mathbb{G}})^{\Gamma(p)})$, where $\Gamma(p) := \text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$.

Remark 5.4.2. Recall that $\mathbb{G}(\mathbb{Q}_{p^r}) \cong (\mathbb{D}_w \otimes_{\mathbb{Q}_p} \mathbb{Q}_{p^r})^\times \times \mathbb{Q}_{p^r}^\times$. We call the image of $N_r \delta$ in $\mathbb{Q}_{p^r}^\times$ its similitude factor. On the other hand, the first factor of $N_r \delta$ has a characteristic polynomial which is a monic polynomial of degree $(\dim_{\mathbb{F}} \mathbb{D})^{\frac{1}{2}}$ with coefficients in $F \otimes_{\mathbb{Q}_p} \mathbb{Q}_{p^r}$ (recall that $\mathbb{D}_w$ has center $\mathbb{F}_w \cong F$). We refer to this simply as the characteristic polynomial of $N_r \delta$. Similarly, for any element in $\mathbb{G}'(\mathbb{Q}_p) \cong \text{GL}_n(F) \times \mathbb{Q}_p^\times$ we have its characteristic polynomial (defined by the first factor) and its similitude factor. In the definition of generalized Kottwitz triples, the condition that $N_r \delta \in \mathbb{G}(\mathbb{Q}_{p^r})$ is stably conjugate to $\gamma'_0 \in \mathbb{G}'(\mathbb{Q}_p)$ simply means that they have the same characteristic polynomial and similitude factor.

Let $x = (A_\bullet, \lambda, \bar{\eta}) \in \text{Fix}_{j, \mathcal{L}}(g^p)$ be a fixed point in the closure of the generic fiber as in §4.4.2. Then we have associated to x a polarized virtual $\mathcal{O}_{\mathbb{B}}$ -abelian variety (A, u, λ) over $\mathbb{F}_{p^r}$, giving rise to a conjugacy class $\gamma \in \mathbb{G}(\mathbb{A}_f^p) = \mathbb{G}'(\mathbb{A}_f^p)$ and a σ -conjugacy class $\delta \in \mathbb{G}(\mathbb{Q}_{p^r})$.

Lemma 5.4.3. *There exists a unique semi-simple stable conjugacy class $\gamma'_0 \in \mathbb{G}'(\mathbb{Q})$ such that $(\gamma'_0; \gamma, \delta)$ forms a degree j generalized Kottwitz triple.*

Proof. The arguments in [Kot92b, §14] go through in our situation with slight modifications.

First we construct a candidate $\gamma'_0 \in \mathbb{G}'(\mathbb{Q})$. Starting with a maximal semisimple subalgebra N of $\text{End}_{\mathbb{B}}(A)$ containing the Frobenius endomorphism π_A , we seek an algebra embedding from N into $C' := \text{End}_{\mathbb{B}'}(\mathbb{V}') = (\mathbb{B}')^{\text{op}}$ compatible with the Rosati involution $*_\lambda$ on N and the involution $*'$ on C' , and take γ'_0 to be the image of π_A^{-1} under this embedding. To show the existence of such an embedding, one replaces the algebra C in *loc.cit.* by our C' everywhere in pages 420-422. This works in our setting because $\mathbb{B}'$ and hence C' splits at any place of $\mathbb{F}$ above p .

Next we check that $(\gamma'_0; \gamma, \delta)$ forms a generalized Kottwitz triple. For each $\ell \neq p$, over the algebraic closure $\overline{\mathbb{Q}_\ell}$ we have an isomorphism $\mathbb{G}'(\overline{\mathbb{Q}_\ell}) \cong \text{GL}_n(\overline{\mathbb{Q}_\ell})^{\text{Hom}(\mathbb{F}_0, \overline{\mathbb{Q}_\ell})} \times \overline{\mathbb{Q}_\ell}^\times$. Under this isomorphism, each GL_n factor of γ_ℓ and γ'_0 have the same characteristic polynomial, which is the image of the characteristic polynomial of π_A^{-1} in the semisimple $\mathbb{F}$ -algebra N under suitable embeddings of $\mathbb{F}$ into $\overline{\mathbb{Q}_\ell}$. Also they have the same similitude factor in $\overline{\mathbb{Q}_\ell}^\times$, which equals to $(\pi_A \pi_A^{*\lambda})^{-1}$. Hence γ and γ'_0 are stably conjugate in $\mathbb{G}'(\mathbb{Q}_\ell) = \mathbb{G}(\mathbb{Q}_\ell)$.

At p , it is clear that the naive norm $N_r \delta \in \mathbb{G}(\mathbb{Q}_{p^r})$ and $\gamma'_0 \in \mathbb{G}'(\mathbb{Q}_p)$ have the same similitude factor, which equals to $(\pi_A \pi_A^{*\lambda})^{-1}$. Next we recall that there is an $\mathbb{B} \otimes_{\mathbb{Q}} \mathbb{Q}_{p^r}$ -module H_r such that $H_r \otimes_{\mathbb{Q}_{p^r}} \overline{\mathbb{Q}_p}$ is isomorphic to the covariant rational Dieudonné module of the p -divisible group $A[p^\infty]$. The element $\delta \in \mathbb{G}(\mathbb{Q}_{p^r})$ is defined by choosing an isomorphism of $\mathbb{B} \otimes_{\mathbb{Q}} \mathbb{Q}_{p^r}$ -modules $H_r \cong \mathbb{V} \otimes_{\mathbb{Q}} \mathbb{Q}_{p^r}$ (which exists by the assumption that the fixed point $x = (A_\bullet, \lambda, \bar{\eta})$ lies in the closure of the generic fiber). Under the natural embedding $\text{End}_{\mathbb{B}}(A) \rightarrow \text{End}_{\mathbb{B}}(H_r) \cong \text{End}_{\mathbb{B}}(\mathbb{V} \otimes_{\mathbb{Q}} \mathbb{Q}_{p^r})$ we have $N_r \delta = \pi_A^{-1}$.

So the characteristic polynomials of $N_r\delta$ and γ'_0 both equal to the characteristic polynomial of π_A^{-1} in the semisimple $\mathbb{F}$ -algebra N . Hence $N_r\delta$ and γ'_0 are stably conjugate.

Finally, the fact that $\kappa_{\mathbb{G}_{\mathbb{Q}_p}}(p\delta) = -\mu^\#$ follows from the determinant condition for the fixed point $x = (A_\bullet, \lambda, \bar{\eta})$. $\square$

5.4.3. Virtual abelian varieties constructed from generalized Kottwitz triples. As a converse to Lemma 5.4.3, starting from a generalized Kottwitz triple $(\gamma'_0; \gamma, \delta)$, we will construct a c -polarized virtual $\mathbb{B}$ -abelian variety up to isogeny. The crucial ingredient will be Lemma 5.4.4, which implies vanishing of the Kottwitz invariant in our setting.

Let I_0 be the centralizer of γ'_0 in $\mathbb{G}'$. It is a connected reductive group over $\mathbb{Q}$ with simply connected derived group. As usual, we let $\hat{I}_0$ be the neutral component of the Langlands dual group of I_0 and put $\Gamma := \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. Recall that we have canonical Γ -equivariant isomorphism $Z(\hat{\mathbb{G}}') = Z(\hat{\mathbb{G}})$, which leads to a canonical Γ -equivariant embedding $Z(\hat{\mathbb{G}}) \subset Z(\hat{I}_0)$.

Using Honda-Tate theory, one constructs from γ'_0 a c -polarizable virtual $\mathbb{B}$ -abelian variety (A, i) over $\mathbb{F}_{p^r}$ up to isogeny as in [Kot92b, p.438-439], with slight modifications. More precisely, let $f \in \mathbb{F}[T]$ be the characteristic polynomial of $(\gamma'_0)^{-1} \in \mathbb{B}'$. For any root α of f in $\bar{\mathbb{F}}$, one first checks that α is a c -number in the sense of *loc. cit.* page 405, paragraph above Lemma 10.4, where $c = (\gamma'_0 \gamma_0^*)^{-1}$. Then one checks that the multiplicity $m(\alpha)$ of the root α is divisible by $(\dim_{\mathbb{F}} \mathbb{B})^{\frac{1}{2}}$ and the quotient annihilates certain invariants at each place v of $\mathbb{F}[\alpha]$. The arguments on *loc. cit.* page 438 go through in our slightly modified setting. Indeed, if v is a place above a prime $\ell \neq p$, then we have $\mathbb{B}' \otimes_{\mathbb{F}} \mathbb{F}[\alpha]_v \cong \mathbb{B} \otimes_{\mathbb{F}} \mathbb{F}[\alpha]_v$; whereas if v divides p , we use that $N_r\delta \in \mathbb{G}(\mathbb{Q}_{p^r})$ has the same characteristic polynomial as $\gamma'_0 \in \mathbb{G}'(\mathbb{Q}_p)$. Finally if v is archimedean, then v is complex and there is nothing to check.

Let π_A be the Frobenius endomorphism of the virtual abelian variety A . For each prime $\ell \neq p$, the $\mathbb{B}[\pi_A]$ -modules $H_\ell := H_1(A, \mathbb{Q}_\ell)$ and $V_\ell := V \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ are isomorphic where π_A acts on V_ℓ by $\gamma_\ell^{-1} \in \mathbb{G}(\mathbb{Q}_\ell)$.

Choose a c -polarization λ of (A, i) . Let $I := \text{Aut}(A, i, \lambda)$. By the same reasoning as in [Kot92b], last paragraph on page 423, I is an inner form of I_0 so that we have a canonical Γ -equivariant isomorphism $Z(\hat{I}) = Z(\hat{I}_0)$.

Next we would like to adjust the polarization λ by a class in $H^1(\mathbb{Q}, I)$ so that the resulting c -polarized virtual B -abelian variety gives rise to the generalized Kottwitz triple $(\gamma'_0; \gamma, \delta)$. This is done by constructing local classes in $H^1(\mathbb{Q}_v, I)$ at each place v that “measure the discrepancy between (A, i, λ) and (γ, δ) at v ” and show that this collection of local classes comes from a global class.

First consider a prime $\ell \neq p$. The c -polarization λ induces a $*$ -alternating form $(\cdot, \cdot)_\lambda$ on the $B[\pi_A]$ -module H_ℓ , where we extend the involution $*$ to $B[\pi_A]$ by requiring $\pi_A^* = c\pi_A^{-1}$. By [Kot92b, Lemma 10.7] we know that $I_{\mathbb{Q}_\ell}$ is the automorphism group of the skew-Hermitian $B[\pi_A]$ -module H_ℓ , where automorphisms are only required to preserve the form $(\cdot, \cdot)_\lambda$ up to a scalar in $\mathbb{Q}_\ell^\times$. Fix an isomorphism of $B[\pi_A]$ -modules $\varphi_\ell : H_\ell \xrightarrow{\sim} V_\ell$ where π_A acts on V_ℓ by γ_ℓ^{-1} . It takes the $*$ -Hermitian pairing $(\cdot, \cdot)$ on V to a $*_\lambda$ -Hermitian pairing $\varphi_\ell^*(\cdot, \cdot)$ on H_ℓ . There exists $x_\ell \in \text{Aut}_{B[\pi_A]}(H_\ell \otimes \bar{\mathbb{Q}}_\ell)$ such that $x_\ell^* \varphi_\ell^*(\cdot, \cdot) = (\cdot, \cdot)_\lambda$. For each $\theta \in \text{Gal}(\bar{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$, let $a_\ell(\theta) := x_\ell \theta(x_\ell)^{-1}$. Then a_ℓ defines a class in $H^1(\mathbb{Q}_\ell, I)$ and its image under the map

$$\text{Inv}_\ell : H^1(\mathbb{Q}_\ell, I) \rightarrow X^*(Z(\hat{I}_0)^{\Gamma_\ell})$$

equals

$$\text{Inv}_\ell(a_\ell) = \alpha_\ell(\gamma'_0; A, \lambda, i) - \alpha_\ell(\gamma'_0; \gamma, \delta).$$

Here $\alpha_\ell(\gamma'_0; A, \lambda, i)$ is the invariant defined in [Kot92b, Page 424] which measures the difference between the skew-Hermitian $\mathbb{B}[\pi_A]$ -modules H_ℓ and V_ℓ where π_A acts on V_ℓ by $(\gamma'_0)^{-1}$; while $\alpha_\ell(\gamma'_0; \gamma, \delta)$ is the invariant defined in [Kot90, §2] that measures the difference between the conjugacy classes γ_ℓ and γ'_0 inside the same stable conjugacy class of $\mathbb{G}(\mathbb{Q}_\ell)$.

Next consider the prime p . The covariant rational Dieudonné module of A defines a skew-Hermitian isocrystal (H_r, Φ) over $\mathbb{Q}_{p^r}$ where Φ is the inverse of the Verschiebung operator. By

[Kot92b, Lemma 10.8], $I_{\mathbb{Q}_p}$ is isomorphic to the automorphism group of (H, Φ) and hence is an inner form of products of Weil restrictions of general linear groups by our assumption that p splits in the imaginary quadratic extension $\mathbb{K}$. Hence $H^1(\mathbb{Q}_p, I) = 1$ and we let a_p be the trivial class.

The following observation of Kottwitz is the key to finding the desired global cohomology class $H^1(\mathbb{Q}, I)$.

Lemma 5.4.4. *We have $Z(\hat{I}_0)^\Gamma = Z(\hat{\mathbb{G}})^\Gamma$.*

Proof. This is contained in the proof of [Kot92a, Lemma 2]. $\square$

Corollary 5.4.5. *There exists a cohomology class $a \in \ker(H^1(\mathbb{Q}, I) \rightarrow H^1(\mathbb{R}, I))$ such that for all finite primes v of $\mathbb{Q}$, its image in $H^1(\mathbb{Q}_v, I)$ equals to the class a_v defined as above.*

Proof. By Lemma 5.4.4, we have $Z(\hat{I}_0)^\Gamma = Z(\hat{\mathbb{G}})^\Gamma$. Moreover, since $a_p = 1$, to show the existence of the class a , it suffices to show that the sum over all finite primes $\ell \neq p$ of the restrictions of $\text{Inv}(a_\ell)$ to $X^*(Z(\hat{\mathbb{G}})^\Gamma)$ is trivial. As in [Kot92b], we can choose a maximal CM subfield N of $\mathbb{B}$ that also embeds into $\text{End}_{\mathbb{B}}(A)$. Under a suitable isomorphism $\mathbb{B} \otimes_{\mathbb{F}} N \cong M_n(N)$, the c -polarized virtual B -abelian variety (A, i, λ) over $\mathbb{F}_{p^r}$ is decomposed as $A = A_0^n$ where A_0 is a c -polarised abelian variety with CM by N . As in [Kot92b, Lemma 13.2], by lifting CM abelian varieties to characteristic 0 we can prove the vanishing of $\sum_{\ell \neq p} \alpha_\ell(\gamma'_0, A_0)|_{Z(\hat{\mathbb{G}})^\Gamma}$, which implies the vanishing of $\sum_{\ell \neq p} \alpha_\ell(\gamma'_0; A, \lambda, i)|_{Z(\hat{\mathbb{G}})^\Gamma}$. On the other hand, by definition, $\alpha_\ell(\gamma'_0; \gamma, \delta)$ is trivial on $Z(\hat{\mathbb{G}})$ since by construction (cf. [Kot90, §2]) it belongs to $\ker(H^1(\mathbb{Q}_\ell, I_0) \rightarrow H^1(\mathbb{Q}_\ell, \mathbb{G}'))$. Thus we get the desired vanishing statement. $\square$

We represent the cohomology class $a \in \ker(H^1(\mathbb{Q}, I) \rightarrow H^1(\mathbb{R}, I))$ by a cocycle of the form $x^{-1}\theta(x)$ for all $\theta \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, where $x \in \text{End}_{\mathbb{B}}(A) \otimes \overline{\mathbb{Q}}$. From the proof of [Kot92b, Lemma 17.1] we see that some scalar multiple λ'' of $\lambda' := \hat{x}\lambda x$ defines a c -polarization of the virtual $\mathbb{B}$ -abelian variety (A, i) and the generalized Kottwitz triple associated to (A, i, λ'') is $(\gamma'_0; \gamma, \delta)$.

5.5. Finishing the proof of the main local theorem. For each generalized Kottwitz triple $(\gamma'_0; \gamma, \delta)$, choose a polarized virtual $\mathcal{O}_{\mathbb{B}}$ -abelian variety (A, λ, i) that maps to it as in §5.4.3. Let $I = I_{(A, \lambda, i)}$ be the automorphism group of (A, λ, i) . Define the number $c(\gamma'_0; \gamma, \delta)$ to be the product of $\text{vol}(I(\mathbb{Q}) \backslash I(\mathbb{A}_f))$ and the cardinality of the finite set

$$\ker[\ker^1(\mathbb{Q}, I_0) \rightarrow \ker^1(\mathbb{Q}, \mathbb{G}')]$$

By the discussion in [Kot90, §3], the (isomorphism class of the) group I is independent of the choice of (A, λ, i) and thus $c(\gamma'_0; \gamma, \delta)$ is well-defined.

Theorem 5.5.1. *The following equality holds*

$$\text{Tr}(\tau \times h f^p | H_{\beta, \xi}^*) = \sum_{(\gamma'_0; \gamma, \delta)} c(\gamma'_0; \gamma, \delta) O_\gamma(f^p) T O_{\delta\sigma}(\phi_{\tau, h}) \text{tr } \xi(\gamma'_0)$$

where $(\gamma'_0; \gamma, \delta)$ ranges over equivalence classes of generalized Kottwitz triples such that the image of γ'_0 in $\mathbb{G}'(\mathbb{Q}_p)$ is transferred from $\mathbb{G}_\beta(\mathbb{Q}_{p^r})$.

Proof. For each (A, u, λ) we associate a generalized Kottwitz triple $(\gamma'_0, \gamma, \delta)$ and the number of (A, u, λ) associated to a given triple $(\gamma'_0, \gamma, \delta)$ is equal to $\ker[\ker^1(\mathbb{Q}, I_0) \rightarrow \ker^1(\mathbb{Q}, \mathbb{G}')]$. Then the identity follows from Corollary 4.4.2. $\square$

Let $f_{\tau, h}^* \in C_c^\infty(\mathbb{G}'(\mathbb{Q}_p))$ be the stable base change transfer of $\phi_{\tau, h} \in C_c^\infty(\mathbb{G}_\beta(\mathbb{Q}_{p^r}))$. Then for any conjugacy class $\gamma_p \in \mathbb{G}'(\mathbb{Q}_p) \cong \text{GL}_n(F) \times \mathbb{Q}_p^\times$ which is the norm of $\delta \in \mathbb{G}_\beta(\mathbb{Q}_{p^r})$, we have

$$S O_{\gamma_p}(f_{\tau, h}^*) = O_{\gamma_p}(f_{\tau, h}^*) = e(\delta) T O_{\delta\sigma}(\phi_{\tau, h})$$

where $e(\delta) = e(\mathbb{G}_{\beta, \delta\sigma}) \in \{\pm 1\}$ is the Kottwitz sign; see §4.5.3.

Corollary 5.5.2. *We have the following equality*

$$\mathrm{Tr}(\tau \times hf^p | H_{\beta, \xi}^*) = \sum_{\pi'} m(\pi') \mathrm{tr} \pi' (f_{\tau, h}^* f^p f_{\xi, \infty}^{\mathbb{G}'})$$

where the sum runs over automorphic representations π' of $\mathbb{G}'$ and $m(\pi')$ denotes the automorphic multiplicity.

Proof. This follows from Theorem 5.5.1 by the pseudo-stabilization process in §4.5. $\square$

Combining Corollary 5.3.3 and Corollary 5.5.2, we will be able to deduce our main local theorems. We claim that for any irreducible generic representation π'_p of $\mathbb{G}'(\mathbb{Q}_p)$ which is not transferred from $\mathbb{G}_\beta(\mathbb{Q}_p)$, $\mathrm{tr} \pi'_p(f_{\tau, h}^*) = 0$. Since the L^2 -Bernstein variety for $\mathbb{G}_\beta(\mathbb{Q}_p)$ is a union of connected components of the L^2 -Bernstein variety for $\mathbb{G}'(\mathbb{Q}_p)$, the property of not being a transfer from $\mathbb{G}_\beta(\mathbb{Q}_p)$ is Zariski closed. Thus by [Shi12, Proposition 3.1], to prove the claim we may assume that π'_p is the local component of an automorphic representation π' of $\mathbb{G}'$ such that $\mathrm{tr} \pi'_\infty(f_{\xi, \infty}^{\mathbb{G}'}) \neq 0$. Choose $f^p \in C_c^\infty(\mathbb{G}'(\mathbb{A}_f^p))$ such that its trace on $(\pi')^{p, \infty}$ is nonzero and its trace on all other irreducible admissible representations of $\mathbb{G}'(\mathbb{A}_f^p) \cong \mathbb{G}_\beta(\mathbb{A}_f^p)$ vanishes. Then by Corollary 5.5.2, $\mathrm{Tr}(\tau \times hf^p | H_\xi^*)$ is a nonzero multiple of $\mathrm{tr} \pi'_p(f_{\tau, h}^*)$. If this is nonzero, then by Corollary 5.3.3 (with the same choice of f^p), there is an automorphic representation π of $\mathbb{G}_\beta$ such that $\pi^{p, \infty} \cong (\pi')^{p, \infty}$. But then by Theorem 5.2.2 and strong multiplicity one, π and π' have the same transfer to an automorphic representation of $\mathrm{GL}_{n, \mathbb{F}}$ and hence π'_p is the Jacquet-Langlands transfer of π_p , contradicting our assumption at the beginning. This proves our claim and then Theorem 3.4.3 follows by Lemma 2.5.2.

Next we consider an irreducible generic representation π_p of $\mathbb{G}_\beta(\mathbb{Q}_p)$ and let π'_p be its Jacquet-Langlands transfer to $\mathbb{G}'(\mathbb{Q}_p)$. We claim that

$$\mathrm{tr}(z_{\tau, -\mu} * h | \pi_p) = \mathrm{tr}(f_{\tau, h}^* | \pi'_p).$$

We may assume that π_p is the local component of an automorphic representation π of $\mathbb{G}_\beta(\mathbb{A})$ such that $\mathrm{tr} \pi_\infty(f_{\xi, \infty}^{\mathbb{G}}) = 1$. By Theorem 5.2.2, [Bad08, Theorem 5.1] and [HT01, Theorem VI.2.9], π'_p is the local component of an automorphic representation π' of $\mathbb{G}'$ such that $\mathrm{tr} \pi'_\infty(f_{\xi, \infty}^{\mathbb{G}'}) = 1$ and $\pi_f^p \cong (\pi'_f)^p$. We choose f^p that singles out π_f^p as before, then the claim follows from Corollary 5.3.3 and Corollary 5.5.2. Combined with the previous claim we proved (that $\mathrm{tr} \pi'_p(f_{\tau, h}^*) = 0$ if π'_p is not transferred from $\mathbb{G}_\beta(\mathbb{Q}_p)$), we see that $f_{\tau, h}^*$ is a Jacquet-Langlands transfer of $z_{\tau, -\mu} * h$. For any semisimple $\gamma \in \mathbb{G}_\beta(\mathbb{Q}_p) = G(F)$ let $\gamma^* \in G^*(F)$ be a matching element. If $\gamma = \mathcal{N}_r \delta$ and hence $\gamma^* = \mathcal{N}_r^* \delta$ for some $\delta \in \mathbb{G}_\beta(\mathbb{Q}_{p^r}) = G(F_r)$, we have

$$e(G_\gamma) O_\gamma(z_{\tau, \mu} * h) = O_{\gamma^*}(f_{\tau, h}^*) = e(G_{\delta\sigma}) T O_{\delta\sigma}(\phi_{\tau, h}).$$

If γ is not conjugate to a norm from $G(F_r)$, then γ^* is also not a norm from $G(F_r)$ and hence we have

$$e(G_\gamma) O_\gamma(z_{\tau, \mu} * h) = O_{\gamma^*}(f_{\tau, h}^*) = 0.$$

This shows that $z_{\tau, \mu} * h$ is a base change of $\phi_{\tau, h}$ in the sense of Definition 2.1.3, and hence this finishes the proof of Theorem 3.4.4.

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