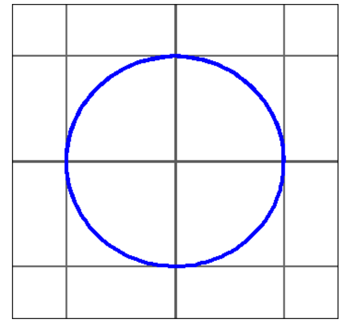
Precalculus 115, section 5.1 The Unit Circle & Radian Measure of Angles

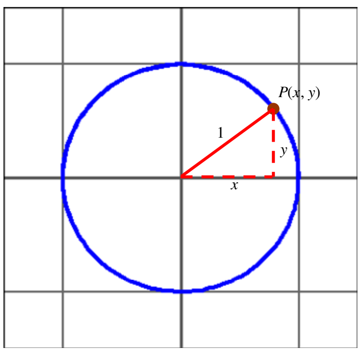
notes by Tim Pilachowski

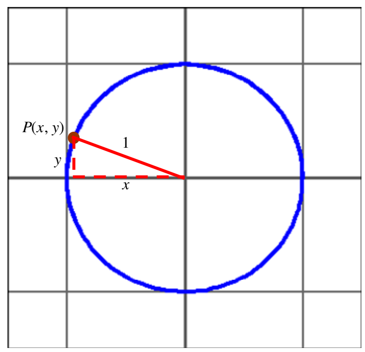


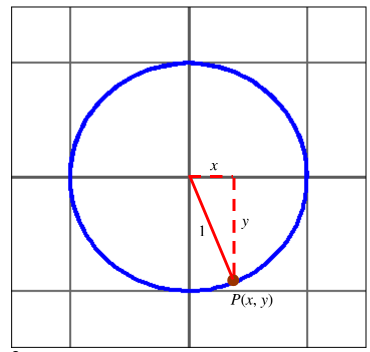
We'll be working with the *unit circle*, with radius equal to 1. On a Cartesian grid we'll place the center of the circle at the origin.

Note first the symmetries of the unit circle: horizontal, vertical and across the origin. If we know coordinates of a point (a, b) in QI, then we know the coordinates of symmetric points in the other three quadrants.

Note that any point P on the unit circle with coordinates (x, y) can be used to form a right triangle, by starting at the origin and moving "over x" and "up/down y" (forming the right angle). The hypotenuse is a radius of the circle.

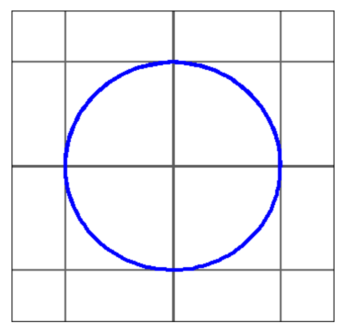




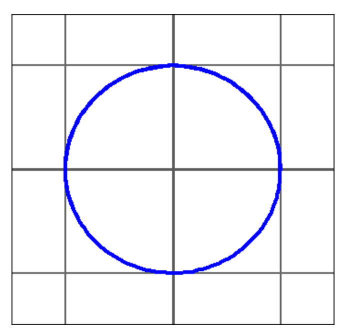


Using the Pythagorean formula, we can derive the equation of the unit circle: $x^2 + y^2 = 1$. We'll be using this basic identity a lot in chapters 5 through 7.

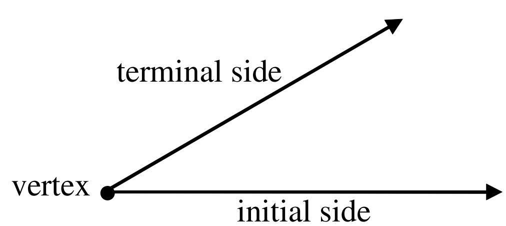
Example A. Find $P(x,y)$ given $x > 0$ and $y = -\frac{\sqrt{11}}{6}$.



Example B. Find $P(x,y)$ given $x = -\frac{2}{3}$ and P in QIII.



We now move to a consideration of *angles*, a geometric and trigonometric figure consisting of a *vertex* and two sides: the *initial side* and the *terminal side*. We'll place the vertex at the center of the unit circle. By convention, the initial side will be drawn horizontally to the right along the x-axis. The terminal side will connect the origin with a terminal point P on the unit circle.



In elementary and high school Geometry, you would likely have measured angles with degrees, minutes and seconds—a method devised in ancient Babylon. Angle measures such as 0º, 30º, 45º, 60º, 90º (right angle), 180º (straight angle), and 360º (full circle) would have been often-used.

For purposes of Trigonometry and Calculus, we'll find it much more convenient to measure angles in terms of *radians.* To define radians, we put our angle into a circle so that the vertex is located at the center, and designate the radius of the circle equal to 1. On a Cartesian grid we'll place the (vertex of the angle) = (center of the circle) at the origin. This *unit circle* has a circumference (formula $C = 2\pi r$) equal to 2π.

Some quick notes on the number π. It is an irrational number, i.e. it cannot be written as an $\frac{\text{integer}}{\text{integer}}$ fraction, and as a decimal it is non-terminating and non-repeating. (See Pilachowski's Rules of Mathematics 3.14 and 3.14159.) It is an *exact* value when written as π—3.14 and $\frac{22}{7}$ are *approximations*. In this class it will be standard procedure to give the exact value as your answer.

Back to the unit circle. Its circumference is equal to 2π. Radian measure is defined as the distance around the circumference one must travel to get from the initial side of the angle to the terminal side. If one travels all the way around the unit circle, it would be an angle of 360º and 2π radians. This relationship gives us the conversion factor to convert degrees to radians and back again:

$$\text{degrees} * \frac{2\pi}{360} = \text{radians} \quad \text{and} \quad \text{radians} * \frac{360}{2\pi} = \text{degrees} \; .$$

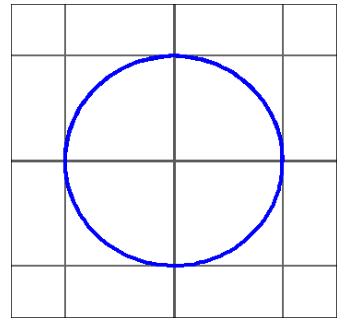
Note that the text uses the simplest form fractions $\frac{\pi}{180}$ and $\frac{180}{\pi}$.

Example C: Convert 0º, 30º, 45º, 60º, 90º and 180º to radians. *answers:* $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}, \pi$; *Note that* $\frac{\pi}{2}$ *and* π *radians, when placed on the unit circle, match what we know about right and straight angles.*

Example D: Convert $\frac{3\pi}{4}$, $\frac{7\pi}{6}$ and $\frac{3\pi}{2}$ radians to degrees. *answers:* 135º, 210º, 270º

Note that in the latter two answers to Example D, we came up with angles not seen in Geometry, i.e angles larger than 180º. (This is just one one of the reasons why radians will be more useful—and convenient—than degrees.)

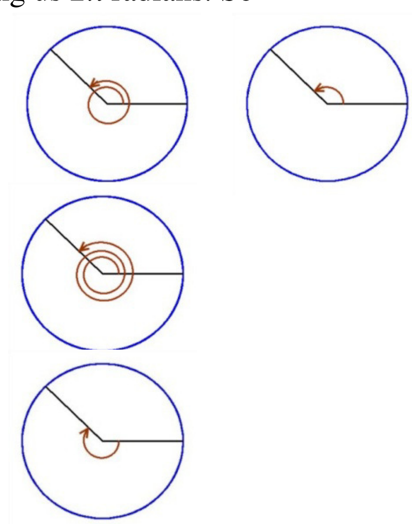
Example E: Find $P(x,y)$ for angle $t = \frac{3\pi}{2}$.



The convention is that we will also designate the direction of travel as follows: *couterclockwise* is positive movement, and *clockwise* is negative movement.

Note: A central angle of $-\frac{\pi}{3}$ radians puts us at the same place as $+\frac{5\pi}{3}$ radians. We'll call these two angles *coterminal*.

We can also travel more than once around the circle, with each transit taking us 2π radians. So

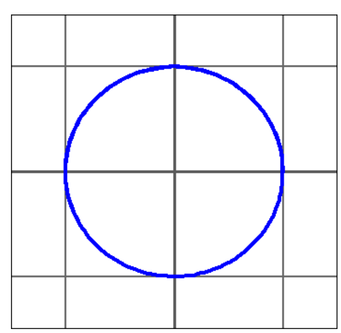


$\frac{11\pi}{4} = \frac{8\pi}{4} + \frac{\pi}{4} = 2\pi + \frac{3\pi}{4}$ is coterminal with $\frac{3\pi}{4}$,

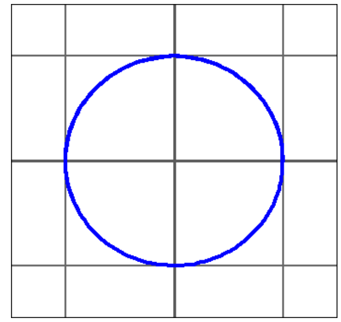
as well as with $\frac{19\pi}{4} = \frac{16\pi}{4} + \frac{3\pi}{4} = 4\pi + \frac{3\pi}{4}$, and so on.

Likewise, $-\frac{5\pi}{4} = \frac{3\pi}{4} - \frac{8\pi}{4} = \frac{3\pi}{4} - 2\pi$ is also coterminal with $\frac{3\pi}{4}$, however it is very important to note that its direction of travel is clockwise rather than counterclockwise.

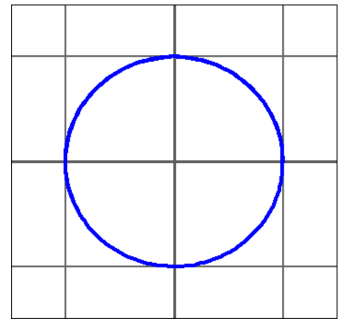
Example E: Find $P(x,y)$ for angle $t = -9\pi$.



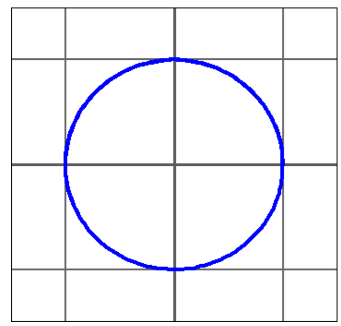
$P(x,y)$ for angle $t = \dfrac{\pi}{6}$



$P(x,y)$ for angle $t = \dfrac{\pi}{3}$



$P(x,y)$ for angle $t = \dfrac{\pi}{4}$

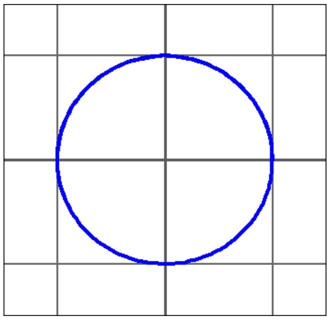


Reference angles. You'll need to know the following values for angles in Quadrant I.

t	x	y
$\dfrac{\pi}{2}$		
$\dfrac{\pi}{3}$		
$\dfrac{\pi}{4}$		
$\dfrac{\pi}{6}$		
0		

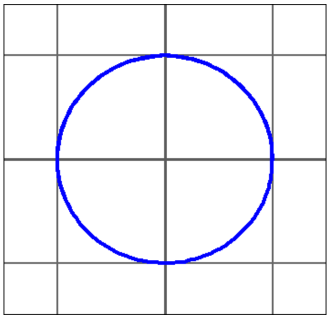
If we know $P(x,y)$ for a reference angle t in QI, then we can find $P(x,y)$ for any angle in the other three quadrants.

$t = \frac{2\pi}{3}$



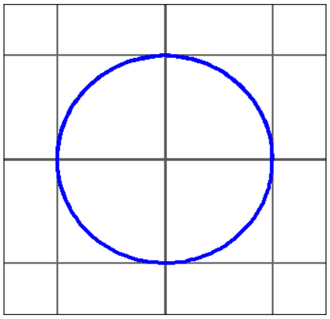
The angle $t = \frac{\pi}{3}$ is also the reference angle for all of the angles symmetric to it in QIII and QIV.

$t = -\frac{\pi}{6}$



The angle $t = \frac{\pi}{6}$ is also the reference angle for all of the angles symmetric to it in QIII and QIV.

$t = \frac{5\pi}{4}$ and $t = -\frac{3\pi}{4}$



The angle $t = \frac{\pi}{4}$ is also the reference angle for all of the angles symmetric to it in QIII and QIV.