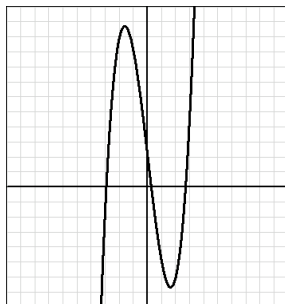


# Calculus 120, section 1.2 The Slope of a Curve at a Point

notes by Tim Pilachowski

Finding the slope of a line is fairly simple, once you get the hang of it, because the slope is the same everywhere on the line. Another way to say the same thing is that, while the *value* of a linear function (such as  $y = 4x + 1$ ) may change, its *rate of change* is constant. [There is only one type of linear function whose *value* doesn't change. What type is that?]



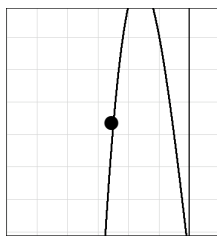
Consider the function  $f(x) = x^3 - 8x + 2$ , pictured to the left. (To reproduce this on your graphing calculator set your window to  $[-10, 10]$  by  $[8, 12]$ .) The curve's "slope" is far from constant. From  $-\infty$  until someplace between  $x = -2$  and  $x = -1$  the function is increasing (has a positive slope). The curve then levels off (slope = 0, a *relative maximum*), changes direction, and is decreasing (slope is negative) until someplace between  $x = 1$  and  $x = 2$ . On the other side of this *relative minimum* (toward  $\infty$ ) the function is once again increasing.

The task at hand will be to find a way to describe the rate of change of  $f(x)$  in a way that not only makes sense intuitively, but also complements all of the other information we know about functions and their graphs. We'll develop a method by traveling along the curve a piece at a time.

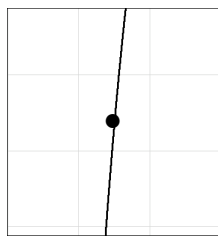
Example A: First step: the open interval  $(-\infty, -2)$ . The series of graphs below zoom in on the point

$\left(-\frac{5}{2}, \frac{51}{8}\right)$ . (You can duplicate this on your graphing calculator. Begin with the window in **Standard**. Use the

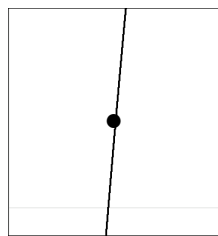
**TRACE** key to get your cursor close to  $x = -2.5$ , press the **ZOOM** key, then choose **Zoom In**. Each time you press **ENTER** the calculator will show pictures similar to those below.) Dimensions of the graph window are given below each picture.



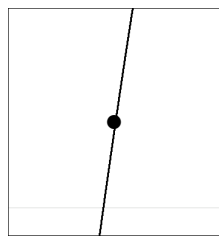
$[-6, 1]$  by  
 $[2.875, 9.875]$



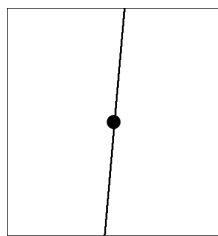
$[-4, -1]$  by  
 $[4.875, 7.875]$



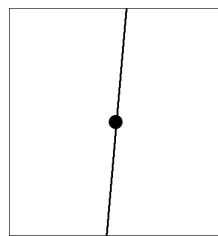
$[-3, -2]$  by  
 $[5.875, 6.875]$



$[-2.8, -2.2]$  by  
 $[6.075, 6.675]$

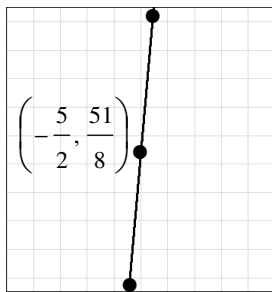


$[-2.6, -2.4]$  by  
 $[6.275, 6.475]$



$[-2.55, -2.45]$  by  
 $[6.325, 6.425]$

We could continue the process, but the pictures would change very little. Notice what happens as we zoom in. It is clear in the original graph that at the point  $\left(-\frac{5}{2}, \frac{51}{8}\right)$  the function has a steep positive slope. As we zoom in, the graph appears to have less curve and is more like a line.



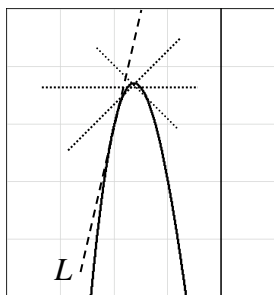
The final picture is reproduced to the left, with a grid marked in intervals of 0.01. We can roughly estimate the slope by approximating points and calculating using the formula for the slope of a line. The point at the top of the picture is approximately  $(-2.495, 6.425)$ .

The point at the bottom of the picture is approximately  $(-2.505, 6.325)$ . We estimate the slope to be  $\frac{6.325 - 6.425}{-2.505 - (-2.495)} = \frac{-0.1}{-0.01} = 10$ . This estimate is close, but is not exact—the

slope at this point is actually 10.75.

Before we move to the next interval on our curve, near the relative maximum, we need to be a little more formal in our definitions: The *slope of a curve at a point P* is defined to be the slope of the line tangent to the curve at point  $P$ .

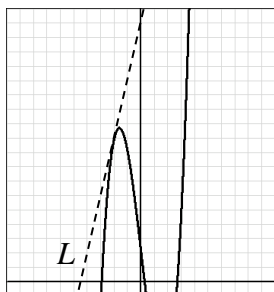
The function  $f(x) = x^3 - 8x + 2$  has a relative maximum someplace in between  $x = -2$  and  $x = -1$ . The lines tangent to the curve vary greatly (see dotted lines in the picture below). How can we estimate the slope of the curve near this point? Also, how can we determine the coordinates of this relative maximum? The answer is that we need some more rigorous mathematics of the kind which we'll derive in the next section. For now, we'll



simply state as a fact that, for the function  $f(x) = x^3 - 8x + 2$ , the slope of the curve at a point  $(x, f(x))$  can be found with the formula  $3x^2 - 8$ .

Example B: Find the slope of the curve  $f(x) = x^3 - 8x + 2$  at  $x = -2$ . *Answer: 4*

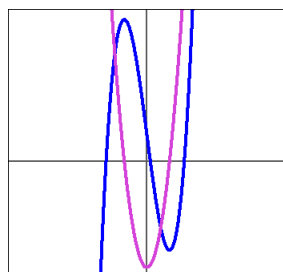
Example B extended: Find the equation of this tangent line. *Answer:  $y = 4x + 18$*



Both  $f$  and its tangent at  $x = -2$  have been graphed in the picture to the left.

Example C: Find the  $x$ -coordinates of the relative maximum and relative minimum of  $f$ . *Answers:  $\pm 2\sqrt{\frac{2}{3}}$*

$x = \pm 2\sqrt{\frac{2}{3}}$  is the *exact* answer. You could use the **TRACE** key and arrows on your graphing calculator, but the answer you get would be only a decimal approximation.



Example D: How does the graph of  $f(x) = x^3 - 8x + 2$  relate to the graph of the “slope of the curve” equation  $f'(x) = 3x^2 - 8$ ?

Both are graphed in the picture to the left. Note that the parabola has positive values on the open interval  $\left(-\infty, -2\sqrt{\frac{2}{3}}\right)$ , corresponding to a positive slope for  $f$ . At  $x = -2\sqrt{\frac{2}{3}}$ ,

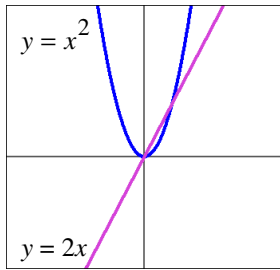
the parabola is at 0, corresponding to the relative maximum on  $f$ . On the open interval

$-2\sqrt{\frac{2}{3}} < x < 2\sqrt{\frac{2}{3}}$  the parabola has negative values, corresponding to a negative slope for  $f$ . At  $x = 0$  the

parabola has a vertex, corresponding to what we'll later call a point of inflection on  $f$ . At  $x = 2\sqrt{\frac{2}{3}}$ , the parabola

is at 0, corresponding to the relative maximum on  $f$ . On the open interval  $\sqrt{\frac{8}{3}} < x < \infty$  the parabola once again

has positive values, corresponding to a positive slope for  $f$ .



Example E: Many of the practice exercises in the text rely on the fact that the [slope of the graph of  $y = x^2$  at the point  $(x, y)] = 2x$ . Note that this formula fits what we see in the graph. When  $x$  is negative, the function is decreasing, and the “slope” is negative. When the curve levels out at  $x = 0$ , the “slope” also equals 0. To the right of the vertex, the function is increasing and the “slope” is positive.

These concepts will become increasingly important as we go along, particularly the notion of the “slope” equaling 0 where a curve “levels out”. Using this idea will provide us the means of determining the precise location of all relative maxima and minima.