Calculus 120, section 1.3 The Derivative

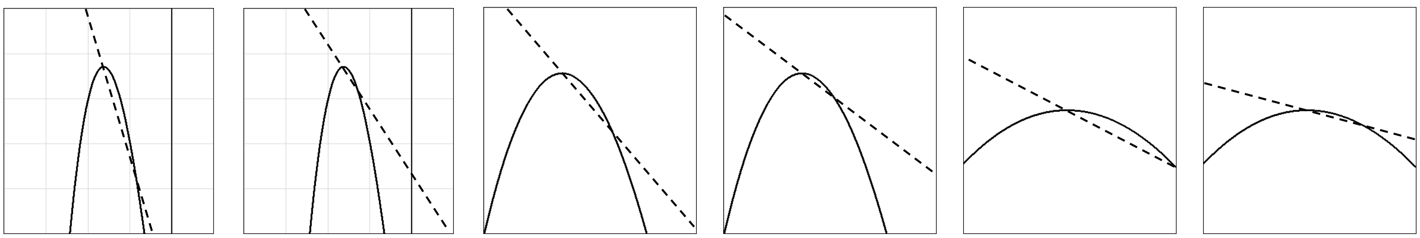
notes by Tim Pilachowski, Spring 2005, revised Fall 2005

We now take the idea of section 1.2 (the slope of a curve at a point *P*) and treat it more formally. Given a function *f*(*x*), the **first derivative of *f*** is a formula which provides the slope of the curve at any point on *f*. Note that the derivative of a function is itself a function. The process of finding a derivative is called **differentiation**.

The first derivative of *f* has several notations: f', $f'(x)$, $\frac{dy}{dx}$, and $\frac{d}{dx}[f(x)]$.

Example A: Given $f(x)=x^3-8x+2$, estimate the derivative evaluated at $x=-\sqrt{\frac{8}{3}}$, $f'\left(-\sqrt{\frac{8}{3}}\right)$.

The pictures below show a series of secant lines connecting the point $\left(-\sqrt{\frac{8}{3}}, f\left(-\sqrt{\frac{8}{3}}\right)\right)$ to other points on *f*.



As the second point approaches $\left(-\sqrt{\frac{8}{3}}, f\left(-\sqrt{\frac{8}{3}}\right)\right)$, the secant lines approach the tangent line. If we continued to zoom in, the secant lines would come closer and closer to horizontal. So our estimate for $f'\left(-\sqrt{\frac{8}{3}}\right)$ is 0.

Example A extended: Given $f(x)=x^3-8x+2$, estimate $f'\left(-\sqrt{\frac{8}{3}}\right)$.

While the visual sequence above seems to point to a particular result, it is far from accurate. A point misplaced by even a little bit will change the picture greatly. We can use decimal approximations to gain a little more precision. In the table below *h* will indicate the amount added to $x=-\sqrt{\frac{8}{3}}$ to place the second point on the curve of *f*. The slope of the secant line thus formed is calculated using the familiar linear slope formula:

$$m=\frac{\Delta y}{\Delta x}=\frac{f\left(-\sqrt{\frac{8}{3}}+h\right)-f\left(-\sqrt{\frac{8}{3}}\right)}{\left(-\sqrt{\frac{8}{3}}+h\right)-\left(-\sqrt{\frac{8}{3}}\right)}=\frac{f\left(-\sqrt{\frac{8}{3}}+h\right)-f\left(-\sqrt{\frac{8}{3}}\right)}{h}$$

$f\left(-\sqrt{8/3}\right)\cong$	$h=$	$-\sqrt{8/3}+h\cong$	$f\left(-\sqrt{8/3}+h\right)\cong$	secant slope $\cong$
10.709296863	1	-0.632993162	6.810317378	-3.898979486
10.709296863	0.1	-1.532993162	10.661307068	-0.479897949
10.709296863	0.01	-1.622993162	10.708807965	-0.048889795
10.709296863	0.001	-1.631993162	10.709291965	-0.004897979
10.709296863	0.0001	-1.632893162	10.709296814	-0.000489888
10.709296863	0.00001	-1.632983162	10.709296863	-0.000048990
10.709296863	0.000001	-1.632992162	10.709296863	-0.000004899
10.709296863	0.0000001	-1.632993062	10.709296863	-0.000000497
10.709296863	0.00000001	-1.632993152	10.709296863	0.000000000

So, with an accuracy to nine decimal places, as the second point approaches $\left(-\sqrt{\frac{8}{3}}, f\left(-\sqrt{\frac{8}{3}}\right)\right)$, the slope of the secant lines approaches 0. So our estimate for $f'\left(-\sqrt{\frac{8}{3}}\right)$ is 0.

Neither of the above methods is sufficient mathematically, because even with the added precision of decimals, we cannot be sure that our answer is *exact*, as opposed to being extremely close. Also, neither method is practical to reproduce for every point on the curve. However, notice first of all that the slope of each sequential secant line is the difference quotient $\frac{f(x+h)-f(x)}{h}$. As h approaches 0, sequence of secant lines approaches the tangent line, and the sequence of slopes approaches the slope of the tangent.

Example A one last time: Given $f(x)=x^3-8x+2$, derive a formula for $f'(x)$ then calculate $f'\left(-\sqrt{\frac{8}{3}}\right)$.

Answers: In mathematical notation: Given $f(x)=x^3-8x+2$, $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=3x^2-8$; 0

Example B: Given $f(x) = x^2$, find $\frac{d}{dx}\left(x^2\right)$. *Answer*: $2x$

We can go through a similar process for any function of the form $f(x)=x^r$ where r could be any number. Specifically, we'd find that $f'(x)=r\,x^{r-1}$. Such a function is called a **power function**, and this property of derivatives is called the **power rule**.

Example C: Given $y = x$, find $\dfrac{dy}{dx}$. *Answer*: 1

Note that the answer to Example C fits with what we already know about lines in slope-intercept form.

Example D: Given $f(x) = 5$, find $f'(x)$. *Answer*: 0

Note that this answer fits with what we already know about horizontal lines.

Example E: Given $f(x) = \sqrt{x}$, find the first derivative of *f*. *Answer*: $\frac{1}{2\sqrt{x}}$.

Example F: Given $y = \dfrac{1}{x^3}$, find y'. *Answer*: $-\frac{3}{x^4}$.

Example G: Given $f(x) = \dfrac{\sqrt[3]{x}}{x^2}$, find $f'(x)$. *Answer*: $-\frac{5}{3x^{8/3}}$.

Some final notions about derivatives and differentiability:

In line with the concept of slope of the curve at a point being equal to the slope of the tangent line at that point, $f'(x) > 0$ implies that *f* is increasing. Likewise $f'(x) < 0$ implies that *f* is decreasing. However, the implication does *not* go the other way. The fact that a function is increasing does *not* imply that $f'(x) > 0$. The simplest example is $y = x^3$. Although x^3 is increasing everywhere on its domain, its derivative is not always positive. The curve momentarily "levels out" precisely when $x = 0$, then continues upward. At this point $\dfrac{d}{dx}(x^3) = 0$. A similar limitation exists for decreasing functions.

Another important concept for differentiability of functions is the idea of being *continuous*. Specifically, the existence of a derivative at a point implies that the function is continuous at that point, i.e. its graph is not disconnected pieces. However, the implication does *not* go the other way. A function may be continuous, but may not be differentiable at all points. Two examples are shown below. On the left, $f(x) = \sqrt{25 - x^2}$ is defined and has values at $x = -5$ and $x = 5$, but the derivative does not exist since the tangent is a vertical line whose slope is undefined. On the right, $f(x) = -|x| + 5$ is defined and continuous for all real numbers, but the derivative does not exist at $x = 0$ because the tangent line on the left has a positive slope, while the tangent line on the right has a negative slope.