Calculus 120, section 3.1 Product Rule and Quotient Rule

notes by Tim Pilachowski

In chapter 2 we found one formula for finding a derivative: the power rule (specific and general versions. Everything else was extensions or applications of the same thing. Now we present two new rules for finding derivatives—these are ones you'll need to memorize.

The first is the *product rule*: $\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x) = fg' + gf'$. The text does a derivation at the end of the chapter. I'll leave it to you to look over that work.

Example A: Given $m(x) = (4x^2 - 1)(3 + x^3)$, find m'.

(In Lecture 1.6 Example D we multiplied this out and used the power rule to find $m'(x) = 20x^4 - 3x^2 + 24x$.)

Example B: Given $n(x) = (5x^4 - 1)^2$, find $n'(x)$.

(In Lecture 1.6 Example E we multiplied this out and used the power rule to find $n' = 200x^7 - 40x^3$.)

Note that we could have written $n' = (5x^4 - 1)(20x^3) + (5x^4 - 1)(20x^3) = 2(5x^4 - 1)(20x^3)$ which is precisely the general power rule applied to $n(x)$.

Example C. Given $f(x) = (2x+1)^4(3x-1)^5$, determine where f has possible extrema or points of inflection. Multiplying these out would be time-intensive and fraught with possible error. The product rule provides an easier method. To avoid confusion of letters, rather than re-use f we'll rewrite in terms of u and v.

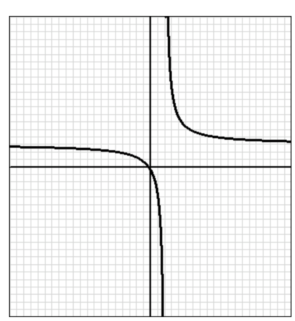
Answer: relative maximum at $x = -\frac{1}{2}$, relative minimum at $x = -\frac{7}{54}$, point of inflection at $x = \frac{1}{3}$

Example D illustrates successive applications of the product rule, with small numbers to be less confusing (hopefully). Example D. Given the polynomial $f(x)=(x+3)(x+1)^2(x-1)^3$, find the first derivative.
Answer: $f'=(x+3)\left[3(x-1)^2(x+1)^2+2(x+1)(x-1)^3\right]+(x+1)^2(x-1)^3$

Example E: Given $y=\left(\sqrt{2x+1}\right)\left(\sqrt{x}-1\right)$ find $\frac{dy}{dx}$. *answer*: $(2x+1)^{1/2}\left(\frac{1}{2}\right)x^{-1/2}+\left(\sqrt{x}-1\right)(2x+1)^{-1/2}=\frac{4x+1-2\sqrt{x}}{2\sqrt{x(2x+1)}}$

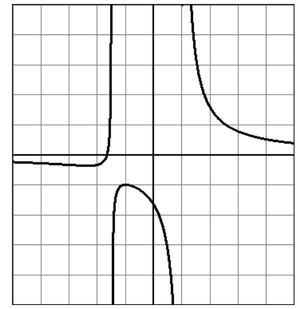
Now we move to the *quotient rule*.
Example F. Prove $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=\frac{g(x)f'(x)-f(x)g'(x)}{[g(x)]^2}=\frac{gf'-fg'}{g^2}$.

Example G: Given $h(x)=\frac{3x+1}{x-2}$ find h'. (See Lecture 2.1.)
Answer: $h'=\frac{-7}{(x-2)^2}$



Example H: Locate the relative maximum of $f(x) = \dfrac{3x+5}{2x^2+x-3}$ between its two vertical asymptotes.

Answer: (–1, –1)



Example I: Average cost (AC) can be thought about as total cost divided by the number of items produced, or $AC = \dfrac{C(x)}{x}$. The text shows that where AC is at its minimum, it is always true that AC = marginal cost (MC).

A small lampshade manufacturer has determined that the cost to produce x lampshades is $C(x) = \frac{1}{8}x^2 + 4x + 200$. Use the quotient rule to find the minimum of AC then show that at this point $AC = MC$.

Answer: $x = 40$ *If we don't get to this one during Lecture, do it yourself for practice using the quotient rule.*

Example J: Given $f(x) = \dfrac{\sqrt[3]{x}}{x^2}$, find $f'(x)$.

In Lecture 1.3 Example G we simplified and then used the power rule to find $f'(x) = -\dfrac{5}{3x^{8/3}}$. *If we don't get to this one during Lecture, do it yourself for practice using the quotient rule.*