Calculus 140, section 2.2 Definition of Limits

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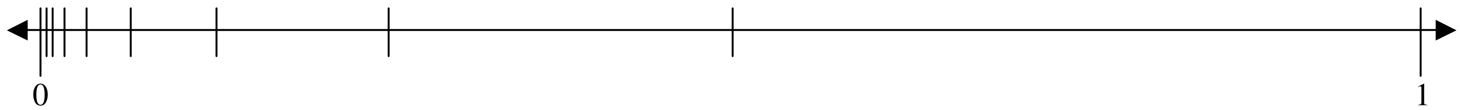
We now move to a more formal examination of the concept of **limits** in mathematics. In section 2.1 we used a non-technical explanation, “moving constantly toward something without ever getting there”. Finding $\lim\limits_{x \to \infty}$ is akin to walking toward the horizon: even though you keep moving, there is always more horizon off in the distance.

Here’s another perspective: Consider decreasing your distance away from an object by half, then half again, then half again, etc. This is like being on a number line at 1, and moving toward 0.



First you’d go to $\frac{1}{2}$, then $\frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \ldots, \frac{1}{2^{100}}, \ldots, \frac{1}{2^{1000}}, \ldots, \frac{1}{2^{1{,}000{,}000}}, \ldots, \frac{1}{2^{1{,}000{,}000{,}000{,}000{,}000}}, \ldots,$

You’d be always getting closer to 0, but never actually reaching 0. In mathematical parlance this would be finding $\lim\limits_{x \to 0^+}$, “the limit as *x* approaches 0 from the right”.

You’ve already encountered limits, albeit in a more limited fashion, when you identified the asymptotes of rational function graphs in an Algebra II or PreCalculus class. Even though you may or may not have called it “taking a limit” at that time, the process was the same. We’ll be a little more formal and rigorous in this class. (You should definitely read through the text’s description and explanation that precedes the definition of limit.)

In Lecture 2.1, the operative phrase was “really, really, really, really close”. This leaves us with the question of how close is close enough?

Definition of Limit: “Let *f* be a function defined at each point of some open interval containing *a*, except possibly *a* itself. Then a number *L* is the **limit of *f*(*x*) as *x* approaches *a*** (or is the **limit of *f* at *a***) if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$\text{if } 0 < \left| x - a \right| < \delta, \text{ then } \left| f(x) - L \right| < \varepsilon \text{”.}$$

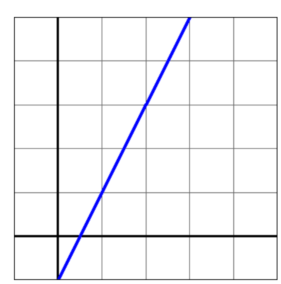
In other words, if we want our *f*(*x*) (i.e. our *y*-value) to come within a distance of ε from *L*, we have to be able to identify a value δ (i.e. how close is close enough?) away from *a* which makes that happen.

If we can, in this way, show that $\lim\limits_{x \to a} f(x) = L$, “then we say that **the limit of *f* at *a* exists**, or that ***f* has a limit at *a***, or that $\lim\limits_{x \to a} f(x)$ **exists**”.

It is implicit from the definition of a limit that, if it exists, the limit of *f* at *a* is unique.

You should definitely take a look at Examples 1 and 2 in the text. They use the definition of a limit to ***prove*** two results that seem intuitively obvious, specifically that $\lim\limits_{x \to a} c = c$ and $\lim\limits_{x \to a} x = a$.

Examples A: Given *f*(*x*) = 2*x* – 1, use the definition of a limit to verify that $\lim\limits_{x \to 1} f(x) =$ ______ .



From section 2.1, we have that the **slope of a line tangent to a graph at a point where *x* = *a*** is

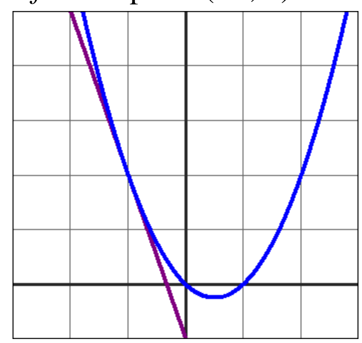
$$m_a = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

1) Note that $\lim_{x \to (x+h)} f(x) = L$ if and only if $\lim_{h \to 0} f(x+h) = L$. This gives us an alternate form for the slope of a line tangent to a graph, $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$, which we will use sometimes when it is more convenient.

2) Substituting into the (I hope) familiar point-slope formula [$y - y_1 = m(x - x_1)$] we get that the **equation of a line tangent to a graph at a point where *x* = *a*** is

$$y - f(a) = m_a(x - a) \quad \text{or} \quad y = f(a) + m_a(x - a).$$

Example B: Given $f(x) = x^2 - x$, find the equation of the line tangent to the graph of *f* at the point (–1, 2).



Next stop (and last stop for section 2.2): velocity. In section 2.1, we determined that, given a function *f*(*t*) that describes position, the velocity at a time *t* (instantaneous velocity) is given by velocity $= \lim_{t \to a} \frac{f(t) - f(a)}{t - a}$. Now in section 2.2, we refine that just a little and consider an object traveling in a straight line at a specific time t_0. The velocity $v(t_0)$ of the object is given by $v(t_0) = \lim_{t \to t_0} \frac{f(t) - f(t_0)}{t - t_0}$.

For Example 4, the text combines this formula for velocity with a result from Precalculus Algebra (see section 1.3) which says that, for an object subject only to the force of gravity, for which distance measured in meters, and time is measured in seconds, the height (position) of the object above ground is given by

$$h(t) = -4.9t^2 + v_0 t + h_0.$$

Specifically for Example 4 (Galileo's experiment of dropping two iron balls from a height of 49 meters) the position function would be $h(t) = -4.9t^2 + 49$. You should read through the rest of Example 4 in the text before trying practice exercises 35 and 36.

Important ideas:

1)

2)

3)

4)