Calculus 140, section 2.3 Limit Rules and Examples

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Recall from Lecture 2.2 – Definition of Limit: "Let f be a function defined at each point of some open interval containing a, except possibly a itself. Then a number L is the **limit of $f(x)$ as x approaches a** (or is the **limit of f at a**) if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$\text{if } 0 < |x-a| < \delta, \text{ then } |f(x) - L| < \varepsilon\text{''.}$$

Here's the good news: We won't have to identify ε and δ every time we want to find a limit. Limits have properties that we will use to make the process much more expedient.

Theorem 2.2: If all of the limits involved exist, then

Sum Rule: $\lim\limits_{x \to a} [f(x) + g(x)] = \lim\limits_{x \to a} f(x) + \lim\limits_{x \to a} g(x)$

Constant Multiple Rule: for any constant c, $\lim\limits_{x \to a} [c * f(x)] = c * \lim\limits_{x \to a} f(x)$

Difference Rule: $\lim\limits_{x \to a} [f(x) - g(x)] = \lim\limits_{x \to a} f(x) - \lim\limits_{x \to a} g(x)$

Product Rule: $\lim\limits_{x \to a} [f(x) * g(x)] = \lim\limits_{x \to a} f(x) * \lim\limits_{x \to a} g(x)$

Quotient Rule: as long as, $\lim\limits_{x \to a} g(x) \neq 0$, $\quad \lim\limits_{x \to a} \left[\dfrac{f(x)}{g(x)}\right] = \dfrac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)}$

[All of these, with more detailed explanations of requirements, are also in your text!]

Bottom lines: The limit of a sum/difference/product is the sum/difference/product of the limits.

For the most part, the limit of a quotient is the quotient of the limits, except when the limit of the denominator equals 0.

Repeated application of Sum and Product Rules give us the limits of polynomial and rational functions (as long as the limit of the denominator does not equal 0.

Example A: Use the properties of limits to find $\lim\limits_{x \to 1} \dfrac{x-1}{x^2+1}$. *answer*: 0

Example B: Use the properties of limits to find $\lim\limits_{x \to 0} \dfrac{[1/(x+1)] - 1}{x}$. *answer*: -1

There are some limits which we will come across in future sections that are very important, and useful.

$$\lim_{x\to a} x^r = a^r \qquad\qquad \lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a} \begin{cases} \text{for all } a \text{ if } n \text{ is odd} \\ \text{for all } a > 0 \text{ if } n \text{ is even} \end{cases}$$

Examples C: Use the properties of limits to find a) $\lim_{x\to 4} \frac{\sqrt{x}-1}{x-1}$ and b) $\lim_{x\to 1} \frac{\sqrt{x}-1}{x-1}$. *answers*: $\frac{1}{3}$, $\frac{1}{2}$

The text also proves limits involving exponential, logarithm and trigonometric functions.

$$\lim_{x\to a} e^x = e^a \qquad \lim_{x\to a} \ln x = \ln a \text{ for } a > 0 \qquad \lim_{x\to a} \sin x = \sin a \qquad \lim_{x\to a} \cos x = \cos a$$

An extremely useful (and important) item involving limits is the Squeezing Theorem.

Theorem 2.3: "Assume that $f(x) \le g(x) \le h(x)$ for all x in some open interval about a except possibly a itself. If $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$, then $\lim_{x\to a} g(x)$ exists and $\lim_{x\to a} g(x) = L$."

The Squeezing Theorem will allow us to evaluate limits that would otherwise be inaccessible. Specifically, we will need $\lim_{x\to 0} \frac{\sin x}{x} = 1$ and $\lim_{x\to 0} \frac{\cos x - 1}{x} = 0$ when we develop the calculus of trigonometric functions.

The text develops a geometric argument to prove $\lim_{x\to 0} \frac{\sin x}{x} = 1$ [Example 7] and an algebraic argument to show that $\lim_{x\to 0} \frac{\cos x - 1}{x} = 0$ [Example 8].

Example D [text exercise #40]: Use the Squeezing Theorem to evaluate $\lim_{x\to 1} \frac{(\ln x)^2}{1+e^{\frac{1}{x-1}}}$.

Hints for text exercises #37 and #39:

1) Trig functions cos x and sin x will always lie between what two values?

2) If we have an inequality such as $-1 \le \cos x \le 1$, and we want to multiply through, why would we want to multiply with positive values?

If we have an inequality and we want to multiply through by a variable such as x, how could we make sure we're multiplying by a positive?

One more rule for limits. The **Substitution Rule** allows us to evaluate limits of composite functions:

$$\lim_{x\to a} g\left(f\left(x\right)\right) = \lim_{y\to c} g\left(y\right) \text{ where } c = \lim_{x\to a} f\left(x\right).$$

The text has a precise statement and proof in the Appendix.

If f is continuous at a, and g is continuous at $f(x)$, then the substitution rule reduces to $\lim_{x\to a} g\left(f\left(x\right)\right) = g\left(f\left(a\right)\right)$.

If the continuity requirement is not met, then we'll have to go the long way around.

Example E. Evaluate $\lim_{x\to 0} \frac{\cos 2x - 1}{x}$.

Example E extended. Evaluate $\lim_{x\to \pi/2} \frac{e^{\sin(2x-\pi)}-1}{\sin(2x-\pi)}$.