Calculus 221, section 11.5a Power Series & Taylor Series

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Example A: Find a series expansion for the function $f(x)=\dfrac{1}{1-x}$. *answer*: $1+x+x^2+x^3+x^4+\ldots$

A series of the form $\sum_{k=0}^{\infty} c_k x^k = c_0 + c_1 x + c_2 x^2 + \ldots$ is called a power series. We'll be most interested in the type which will converge under given circumstances, and will want to specify those conditions. (At the end of the section, the text includes a brief discussion of "radius of convergence".) In particular, we will focus on an extension of the Taylor polynomial for a function into the Taylor series,

$$f(x)=\sum_{k=0}^{\infty} c_k x^k = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$$

Example B: Find a Taylor series expansion for $f(x)$ = cos x. *answer*: $1-\frac{1}{2!}x^2+\frac{1}{4!}x^4-\frac{1}{6!}x^6+\frac{1}{8!}x^8+\ldots$

Example C: Find a Taylor series expansion for the function $f(x) = e^x$. *answer*: $1+x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\frac{1}{4!}x^4+\ldots$

We probably won't have time in the lecture, but for those who are interested, Example Z at the end of this outline demonstrates that, just like $f(x) = e^x$, this Taylor series is also its own derivative and integral (+ C).

Since we can think of power series as "infinite polynomials", one might ask whether they can be differentiated and integrated. Since derivatives and integrals can be done term by term (Sum Rule), we can apply the power rule for derivatives and integrals. Under the specified conditions for convergence,

$$\frac{d}{dx}\left(\sum_{k=0}^{\infty} c_k x^k\right)=\sum_{k=0}^{\infty}\frac{d}{dx}\left(c_k x^k\right) \text{ and } \int\left(\sum_{k=0}^{\infty} c_k x^k\right)=\sum_{k=0}^{\infty}\left(\int c_k x^k\right).$$

We can use these, along with Substitution and the Constant Multiple Rule to derive other Taylor series expansions without having to go through tedious, cumbersome and difficult differentiations. Text exercises have you use Examples A, B and C above. This lecture works only with Example A. Note that all of the following will work only for $|\,x\,| < 1$!

Example D: Find the Taylor series expansion for $f(x)=\frac{20}{1-x}$. *answer*: $20+20x+20x^2+20x^3+20x^4+\ldots$

Example E: Find the Taylor series expansion for $f(x)=\frac{x^5}{1-x}$. *answer*: $x^5++x^6+x^7+x^8+x^9+\ldots$

Example F: Find the Taylor series expansion for $f(x)=\frac{1}{1-x^4}$. *answer*: $1+x^4+x^8+x^{12}+x^{16}+\ldots$

Example G: Find the Taylor series expansion for $f(x)=\frac{1}{1+x}$. *answer*: $1-x+x^2-x^3+x^4+\ldots$

Example H: Find the Taylor series expansion for $f(x)=\dfrac{1}{(1-x)^2}$. *answer*: $1+2x+3x^2+4x^3+5x^4+\dots$

Example I: Find the Taylor series expansion for $f(x) = \ln(1 - x)$. *answer*: $-x-\frac{1}{2}x^2-\frac{1}{3}x^3-\frac{1}{4}x^4-\frac{1}{5}x^5-\dots$

Example J: Find the Taylor series expansion for $f(x) = \ln(1 - x) - 7x$.

answer: $-8x-\frac{1}{2}x^2-\frac{1}{3}x^3-\frac{1}{4}x^4-\frac{1}{5}x^5-\dots$

For some text and WebAssign exercises, you may need to use more than one of the above methods.

Example Z: The text derived the Taylor series for $f(x) = e^x$. Is this Taylor series its own derivative?

For any x, $e^x = \sum_{k=0}^{\infty} \dfrac{x^k}{k!} = 1+x+\dfrac{x^2}{2!}+\dfrac{x^3}{3!}+\dfrac{x^4}{4!}+\dots$

$$\frac{d}{dx}\left(e^x\right)=\frac{d}{dx}\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\dots\right)=0+1+2\left(\frac{x}{2!}\right)+3\left(\frac{x^2}{3!}\right)+4\left(\frac{x^3}{4!}\right)+\dots=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots$$

Also, $\displaystyle\int\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\dots\right)dx = C+x+\frac{1}{2}x^2+\frac{1}{3}\left(\frac{x^3}{2!}\right)+\frac{1}{4}\left(\frac{x^4}{3!}\right)+\dots = C+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\dots$

If we let $x = 0$ and solve, we get $C = 1$.