

Affine structures on surfaces and the twisted cubic cone

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Dedicated to Ravi S. Kulkarni, with admiration

ABSTRACT. We identify the deformation space of marked complete affine structures on the 2-torus \mathbb{T}^2 with the cone over a twisted cubic curve in $\mathbb{R}P^3$.

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Introduction

An affine structure on an n -dimensional manifold is defined by a system of local coordinates where the coordinate changes are locally defined by affine transformations of \mathbb{R}^n . In this way the manifold is locally modeled on an affine space A (corresponding to \mathbb{R}^n , but without the algebraic structure of a vector space). Equivalently, an affine structure on a manifold is a flat torsionfree affine connection. In [12], Kuiper described the (geodesically) complete affine structures on \mathbb{T}^2 and showed they are of two types: either Euclidean structures (*flat Riemannian structures*), or other structures defined by flat but *non-Riemannian* connections. Figures 1 and 2 depict these two types of structures. In this paper we show that they naturally form a *twisted cubic cone* in \mathbb{R}^4 .

In a more geometric context, a *complete affine structure* on a manifold M is a representation of M as a quotient $\Gamma \backslash A$, where A is an affine space, and $\Gamma < \text{Aff}(A)$ a discrete group of affine transformations acting properly on A . More generally, Kuiper classifies affine manifolds covered by *convex domains* in A^2 . He shows that

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if a closed surface admits such a structure then $\chi(M) = 0$, and conjectures this is true without assuming convexity. This was later proved by Benzécri [7].

If Σ is a fixed surface, a *marked affine structure* on Σ is a pair (f, M) where M is an affine manifold and f is diffeomorphism $\Sigma \rightarrow M$. Marked affine structures (f, M) and (f', M') are *equivalent* if they differ (up to isotopy) by an affine isomorphism $M \xrightarrow{\phi} M'$ of affine manifolds such that f is isotopic to $f' \circ \phi$. The space of equivalence classes of marked structures is called the *deformation space*. It has a natural topology (see [10], §6 Classification) which in general can be quite pathological. However, in many cases it is locally homeomorphic to the “finite-dimensional” space $\text{Hom}(\pi_1(\Sigma), G)/G$ where $G = \text{Aff}(A)$.¹ This space is analogous to the Teichmüller space, consisting of equivalence classes of marked Riemann surfaces. The mapping class group $\text{Mod}(\Sigma)$ acts simply transitively on the markings, and its quotient is the Riemann moduli space of Riemann surfaces homeomorphic to Σ .

The classification of affine structures on surfaces was completed by Nagano-Yagi [13] and Arrowsmith-Furness [1]. In particular the deformation space of *all* affine structures on \mathbb{T}^2 is not Hausdorff. In this paper we discuss the subspace $\mathcal{CA}(\mathbb{T}^2)$ corresponding to *complete* structures. Baues [2–4] showed that $\mathcal{CA}(\mathbb{T}^2)$ is homeomorphic² to \mathbb{R}^2 .

A parametrization of this space by a “period map” is given in [6], and raises the question whether this space has a natural *singular* smooth structure. This paper resolves this question.

The deformation space of *marked* complete structures admits a natural action of the *mapping class group* $\text{Mod}(\mathbb{T}^2) \cong \text{GL}(2, \mathbb{Z})$, permuting the markings. Baues observed that this is the natural *linear* action of $\text{GL}(2, \mathbb{Z})$ on \mathbb{R}^2 . This action is chaotic, and the quotient $\text{GL}(2, \mathbb{Z}) \backslash \mathbb{R}^2$ is intractable. Indeed, this dynamical system is orbit-equivalent to the horocycle flow on the unit tangent bundle $\text{SL}(2, \mathbb{Z}) \backslash \text{SL}(2, \mathbb{R})$ of the elliptic modular curve, which is known to be uniquely ergodic.

This dramatically contrasts with the action of $\text{Mod}(\mathbb{T}^2)$ on the deformation space of Euclidean structures. That action is *proper*, with quotient $\mathbb{R}_+ \times \mathfrak{M}_0$ where \mathfrak{M}_0 the Riemann moduli space of elliptic curves and the \mathbb{R}_+ parameter corresponds to the area of a Euclidean structure on a torus. Of course, Euclidean structures on \mathbb{T}^2 are *complete affine* structures. However, since all Euclidean structures are affinely equivalent, the subspace

$$\mathcal{CA}(\mathbb{T}^2)_{\text{Euc}} \subset \mathcal{CA}(\mathbb{T}^2)$$

corresponding to Euclidean structures collapses to a single point in $\mathcal{CA}(\mathbb{T}^2)$. This point is the origin in $\mathbb{R}^2 \approx \mathcal{CA}(\mathbb{T}^2)$, which maps to the unique singular point in the twisted cubic cone.

The main theorem of the paper is the following:

THEOREM. *The deformation space of affine equivalence classes of marked complete affine structures on \mathbb{T}^2 naturally identifies with a twisted cubic cone $\mathfrak{C} \subset \mathbb{R}^4$.*

¹ $\text{Hom}(\pi_1(\Sigma), G)$ has the natural structure as a real affine algebraic set, and $\text{Hom}(\pi_1(\Sigma), G)/G$ is given the quotient topology by the action of $\text{Inn}(G)$ by composition.

²Indeed, for a while it was believed [9] that the subspace comprising *complete* structures is not Hausdorff. Baues [2] addresses this error in the literature.

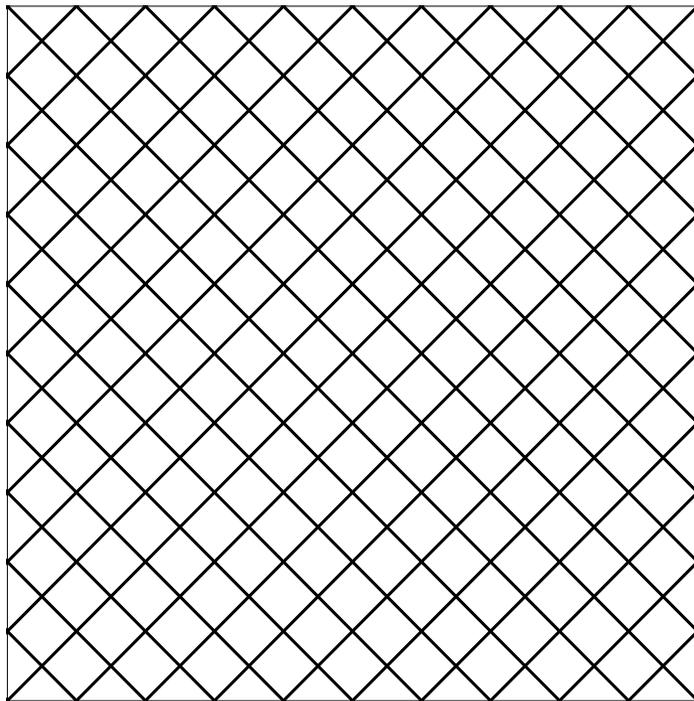


FIGURE 1. One example of a complete affine 2-manifold is a Euclidean flat torus, obtained as the quotient of the Euclidean plane \mathbb{E}^2 by a lattice $\Lambda \in \mathbb{R}^2$ of translations.

Outline of the paper

§1 describes the cone on the twisted cubic and its symmetries. §2 describes the theory of affine structures and reduces the classification to commutative nilpotent 2-dimensional \mathbb{R} -algebras, which are studied in §3. Such an algebra \mathfrak{A} , together with a basis, completely determines the marked structure. From the general theory, this already gives a proof of our main theorem. One key point is that completeness is equivalent to the connection being *equiaffine* (sometimes called *parallel volume*). This means its holonomy preserves volume, that is, its linear holonomy has determinant ± 1 . In this particular case, it is equivalent to the stronger condition of unipotency of the linear holonomy. Compare the discussion of the Markus conjecture in [10], §11 and the proof of completeness of closed affine manifolds with unipotent holonomy in [10], §8.4.

§4 gives a direct proof of the main theorem, using the Christoffel symbols Γ_{ij}^k of the flat torsionfree equiaffine connection, that the deformation space $\mathcal{CA}(\mathbb{T}^2)$ of complete affine structures identifies with the cone \mathfrak{C} on the twisted cubic. The Christoffel symbols are just the structure constants of the algebra \mathfrak{A} . The key point is that Kuiper's classification implies that the identity component of the affine automorphism group acts simply transitively on M . This implies that the Christoffel symbols are constant, enabling the identification of marked structures with nilpotent commutative associative 2-dimensional algebras over \mathbb{R} .

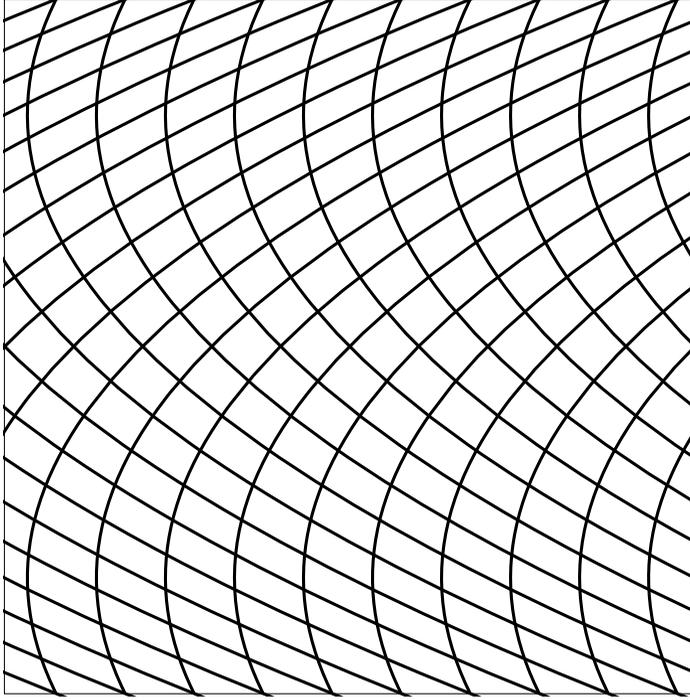


FIGURE 2. Applying a polynomial diffeomorphism of \mathbb{A}^2 such as $(x, y) \xrightarrow{f} (x + y^2, y)$ conjugates translations to a simply transitive affine action of \mathbb{R}^2 which is *not* Euclidean-isometric. The quotient $\mathbb{A}^2/f\Lambda f^{-1}$ is a complete affine 2-torus.

1. The cone on the twisted cubic

The *twisted cubic cone* $\mathfrak{C} \subset \mathbb{R}^4$ is the image of the map

$$(1.1) \quad \mathbb{R}^2 \hookrightarrow \text{Sym}^3(\mathbb{R}^2) \cong \mathbb{R}^4$$

$$\begin{bmatrix} A \\ B \end{bmatrix} \longmapsto \begin{bmatrix} A^3 \\ A^2B \\ AB^2 \\ B^3 \end{bmatrix} =: \begin{bmatrix} U \\ X \\ Y \\ Z \end{bmatrix} \in \mathbb{R}^4.$$

which embeds \mathbb{R}^2 topologically in \mathbb{R}^4 . It is singular, and its unique singularity is the origin. It is the cone over the rational normal curve of degree 3 in \mathbb{P}^3 .

Explicitly, \mathfrak{C} is defined by the three homogeneous quadratic equations

$$\begin{aligned} X^2 &= UY \\ UZ &= XY \\ Y^2 &= XZ \end{aligned}$$

in the coordinates U, X, Y, Z . No two of these equations suffice to define \mathfrak{C} , despite \mathfrak{C} having codimension two in \mathbb{R}^4 .

The map (1.1) is $\mathrm{GL}(2, \mathbb{R})$ -equivariant with respect to the standard action on \mathbb{R}^2 and the induced action on the symmetric power $\mathrm{Sym}^3(\mathbb{R}^2)$. The cone $\mathfrak{C} \subset \mathbb{R}^4$ is invariant under this action of $\mathrm{GL}(2, \mathbb{R})$. In particular this action restricts to $\mathrm{GL}(2, \mathbb{Z}) < \mathrm{GL}(2, \mathbb{R})$, and $\mathrm{GL}(2, \mathbb{Z})$ is isomorphic to the mapping class group $\mathrm{Mod}(\mathbb{T}^2)$.

Later in this paper we identify \mathfrak{C} with the deformation space $\mathcal{CA}(\mathbb{T}^2)$ of marked complete affine structures on \mathbb{T}^2 and the $\mathrm{GL}(2, \mathbb{Z})$ -action on \mathfrak{C} with the $\mathrm{Mod}(\mathbb{T}^2)$ -action on $\mathcal{CA}(\mathbb{T}^2)$.

2. Affine structures on the torus

Kuiper [12] showed that every complete affine structure on the torus \mathbb{T}^2 arises as a flat Euclidean torus $\Lambda \backslash \mathbb{E}^2$ where $\Lambda < \mathbb{R}^2$ is a lattice of translations or a quotient $\Gamma \backslash \mathbb{A}^2$ where $\Gamma < G$ is a lattice where G is the group of affine transformations

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 1 & 2b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} a + b^2 \\ b \end{bmatrix}$$

of \mathbb{A}^2 , where $a, b \in \mathbb{R}$. The group G is isomorphic to \mathbb{R}^2 ; indeed is conjugate to the translation subgroup $\mathbb{R}^2 < \mathrm{Aff}(\mathbb{A}^2)$ by the diffeomorphism $(x, y) \mapsto (x + y^2, y)$.

Since the action of G on \mathbb{A}^2 is simply transitive, it induces a left-invariant³ affine structure on G . Furthermore this passes down to an invariant affine structure on the Lie group $\Gamma \backslash G$, so the complete affine structures on \mathbb{T}^2 are all invariant structures on \mathbb{T}^2 with respect to its structure as an abelian Lie group.

Left-invariant affine structures on Lie groups form a rich algebraic theory described by (possibly non-associative) algebras, where the algebraic structure is defined by covariant differentiation of left-invariant vector fields. Commutator in this algebra is just the usual Lie bracket of left-invariant vector fields (because the connection is torsionfree). Flatness of the connection implies the defining condition that the associator is symmetric in its first two arguments; hence these algebras are called *left-symmetric algebras*. Bi-invariant affine structures correspond to the case when this algebra is associative.⁴

Completeness for these structures is equivalent to several conditions, such as the parallelism of right-invariant volume forms, or unipotency of the affine action by left-multiplication

The theory of affine structures on closed manifolds with nilpotent holonomy is studied in [8] (building on Smillie [14] for abelian holonomy). In particular the equivalence of geodesic completeness with parallel volume (or *equiaffinity*) and unipotent linear holonomy is shown there, and expounded in §11 of [10] in a broader context. For left-invariant affine structures on Lie groups — that is, for *affine Lie groups*, — completeness is equivalent to right-invariant volume forms being parallel [11]. Thus for *abelian* affine Lie groups, completeness is equivalent to equiaffinity.

As mentioned in the introduction, \mathbb{T}^2 is the only closed orientable surface supporting an affine structure. Furthermore, a complete affine structure on \mathbb{T}^2 is

³Since G is abelian, it is also right-invariant.

⁴See §10 of [10] for an extensive discussion of this theory.

necessarily invariant. Equivalently, every complete affine structure on \mathbb{T}^2 is invariant under a structure of \mathbb{T}^2 as an *abelian Lie group*.⁵ Baues [3, 4] surveys the classification of affine structures on surfaces in detail; see also [10].

Since the connection is invariant, the covariant derivative of invariant vector fields is invariant, and the *Christoffel symbols*, the coefficients of the covariant derivatives of a basis of invariant vector fields are constant.

3. Commutative nilpotent algebras

THEOREM 3.1. *A marked complete affine structure on \mathbb{T}^2 corresponds to a based 2-dimensional vector space \mathbb{V} together with a symmetric bilinear form Γ making \mathbb{V} into an algebra \mathfrak{A} , such that \mathfrak{A} is a commutative \mathbb{R} -algebra with $\mathfrak{A}^3 = 0$.*

For an extensive discussion, see [10], §8.4. For the reader's convenience we sketch the proof here.

PROOF. Let $\mathbb{T}^2 \rightarrow M$ be a marked complete affine structure, that is a homeomorphism (defined up to isotopy) onto a complete affine 2-manifold M . Choose a universal covering $\widetilde{M} \rightarrow M$ and a developing pair (dev, h) where the developing map $tM \xrightarrow{\text{dev}} \mathbb{A}^2$ is a diffeomorphism and the holonomy homomorphism h is an embedding of $\pi_1(M)$ onto a discrete subgroup $\Gamma < \text{Aff}(\mathbb{A}^2)$ acting properly on \mathbb{A}^2 . By the general algebraic theory described above (or direct calculation as in Kuiper's original classification), Γ is conjugate to a lattice in a subgroup G_μ of $\text{Aff}(\mathbb{A}^2)$ comprising elements having the form

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 1 & 2\mu b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} a + \mu b^2/2 \\ b \end{bmatrix}$$

where $a, b \in \mathbb{R}$ and $\mu \in \mathbb{R}$ is a fixed parameter. The Lie algebra of G is generated by affine vector fields:

$$\begin{aligned} X &:= \partial/\partial x \\ Y_\mu &:= \mu y \partial/\partial x + \partial/\partial y. \end{aligned}$$

Under covariant differentiation these vector fields span an algebra with multiplication given by Table 1.

TABLE 1. A commutative 2-dimensional algebra with $\mathfrak{A}^3 = 0$

	X	Y_μ
X	0	0
Y_μ	0	μX

Furthermore every 2-dimensional commutative associative algebra \mathfrak{A} with $\mathfrak{A}^3 = 0$ has a basis as above: If $\mathfrak{A}^2 = 0$, the multiplication is trivial and we may take any basis X, Y (and $\mu = 0$). Otherwise nilpotence implies that $\dim \mathfrak{A}^2 = 1$. Let X base \mathfrak{A}^2 , and extend $\{X\}$ to a basis $\{X, Y\}$ of \mathfrak{A} . \square

⁵This was known to Kuiper [12], although he didn't state it in this form.

In practice \mathbb{V} will be the abelian Lie algebra of left-invariant (right-invariant) vector fields, and Γ is the covariant derivative operator

$$\begin{aligned} \mathbb{V} \times \mathbb{V} &\xrightarrow{\Gamma} \mathbb{V} \\ (\xi, \eta) &\longmapsto \nabla_{\xi}\eta. \end{aligned}$$

The above theory implies that \mathfrak{A} is nilpotent, and indeed $\mathfrak{A}^3 = 0$ (so associativity is obvious).

Such a connection can be described in terms of the twisted cubic cone (1.1) as follows.

The connection (or the algebra structure) relates to the symmetric power $\text{Sym}^3(\mathbb{V})$ as follows. For convenience choose a nonzero element ω in the line $\bigwedge^2(\mathbb{V})$. A vector $v \in \mathbb{V}$ determines a covector $\omega_v \in \mathbb{V}^*$ by

$$u \wedge v = \omega_v(u)\omega.$$

An endomorphism $\mathcal{E} \in \text{End}(\mathbb{V})$ of rank one is determined by nonzero vectors k (generating the kernel) and j (generating the image) by

$$\mathcal{E} = \omega_k \otimes j.$$

The endomorphism \mathcal{E} is nilpotent if and only if j and k are linearly dependent. In our case, unipotence of the linear action is equivalent to nilpotence of \mathcal{E} .

A connection is given by the tensor product $\psi \otimes \mathcal{E}$ where $\psi \in \mathbb{V}^*$ is a covector and \mathcal{E} is an endomorphism. Flatness of the corresponding connection means that in the decomposition $\mathcal{E} = \omega_v \otimes w$, the covectors ω_v and ψ are linearly dependent. Hence a flat connection taking values in nilpotent endomorphisms corresponds to an element of $\mathbb{V}^* \otimes \text{End}(\mathbb{V})$ of the form $\omega_v \otimes \mathcal{E}_v$ where $\mathcal{E}_v := \omega_v \otimes v$, that is in the image of

$$\begin{aligned} \mathbb{V} &\longrightarrow \mathbb{V}^* \otimes \text{End}(\mathbb{V}) \\ v &\longmapsto \omega_v \otimes (\omega_v \otimes v) \end{aligned}$$

which is evidently equivalent to (1.1). The final section §4 gives a direct identification using only the assumption of area-preserving holonomy rather than unipotent holonomy.

4. Flat torsionfree equiaffine connections

Choose a coordinate system (x^1, x^2) on \mathbb{A}^2 and coordinate vector fields

$$\partial_i := \frac{\partial}{\partial x^i}$$

for $i = 1, 2$. Write ∇_i for the covariant differential operator ∇_{∂_i} and Γ_{ij}^k for the *Christoffel symbols* for ∇ with respect to the frame ∂_i :

$$\nabla_i \partial_j = \Gamma_{ij}^k \partial_k$$

(Einstein summation). If ∇ has zero torsion, $\nabla_i \partial_j = \nabla_j \partial_i$, and

$$(4.1) \quad \Gamma_{21}^k = \Gamma_{12}^k.$$

for $k = 1, 2$. Since ∇ is *equiaffine* $\nabla(dx^1 \wedge dx^2) = 0$. It follows that for every vector field ξ , the curvature operator $\eta \longmapsto \nabla_{\xi}\eta$ has trace zero. Take $\xi = \partial_i$ for $i = 1, 2$ to obtain:

$$(4.2) \quad \Gamma_{i1}^1 + \Gamma_{i2}^2 = 0$$

for $i = 1, 2$. In the following calculations, (4.2) and (4.1) imply

$$(4.3) \quad \begin{aligned} \Gamma_{21}^1 &= \Gamma_{12}^1 = -\Gamma_{22}^2 \\ \Gamma_{21}^2 &= \Gamma_{12}^2 = -\Gamma_{11}^1 \end{aligned}$$

so we reduce our calculations to the four variables

$$\Gamma_{11}^1, \Gamma_{11}^2, \Gamma_{22}^1, \Gamma_{22}^2.$$

Since the covariant derivatives are constant,

$$(4.4) \quad \begin{aligned} \nabla_1 \nabla_2 \partial_1 &= \nabla_1 (\Gamma_{21}^1 \partial_1 + \Gamma_{21}^2 \partial_2) \\ &= (\Gamma_{11}^1 \Gamma_{21}^1 + \Gamma_{12}^1 \Gamma_{21}^2) \partial_1 + (\Gamma_{11}^2 \Gamma_{21}^1 + \Gamma_{12}^2 \Gamma_{21}^2) \partial_2 \end{aligned}$$

$$(4.5) \quad \begin{aligned} \nabla_2 \nabla_1 \partial_1 &= \nabla_2 (\Gamma_{11}^1 \partial_1 + \Gamma_{11}^2 \partial_2) \\ &= (\Gamma_{21}^1 \Gamma_{11}^1 + \Gamma_{22}^1 \Gamma_{11}^2) \partial_1 + (\Gamma_{21}^2 \Gamma_{11}^1 + \Gamma_{22}^2 \Gamma_{11}^2) \partial_2 \end{aligned}$$

$$(4.6) \quad \begin{aligned} \nabla_1 \nabla_2 \partial_2 &= \nabla_1 (\Gamma_{22}^1 \partial_1 + \Gamma_{22}^2 \partial_2) \\ &= (\Gamma_{11}^1 \Gamma_{22}^1 + \Gamma_{12}^1 \Gamma_{22}^2) \partial_1 + (\Gamma_{11}^2 \Gamma_{22}^1 + \Gamma_{12}^2 \Gamma_{22}^2) \partial_2 \end{aligned}$$

$$(4.7) \quad \begin{aligned} \nabla_2 \nabla_1 \partial_2 &= \nabla_2 (\Gamma_{12}^1 \partial_1 + \Gamma_{12}^2 \partial_2) \\ &= (\Gamma_{21}^1 \Gamma_{12}^1 + \Gamma_{22}^1 \Gamma_{12}^2) \partial_1 + (\Gamma_{21}^2 \Gamma_{12}^1 + \Gamma_{22}^2 \Gamma_{12}^2) \partial_2 \end{aligned}$$

Flatness of ∇ implies $\nabla_1 \circ \nabla_2 = \nabla_2 \circ \nabla_1$. Subtracting (4.5) from (4.4), $[\nabla_1, \nabla_2] \partial_1 = 0$ implies:

$$(4.8) \quad \begin{aligned} \Gamma_{12}^1 \Gamma_{21}^2 &= \Gamma_{22}^1 \Gamma_{11}^2 \\ &\text{for the } \partial_1\text{-component of } [\nabla_1, \nabla_2] \partial_1 \end{aligned}$$

$$(4.9) \quad \begin{aligned} \Gamma_{11}^2 \Gamma_{21}^1 + \Gamma_{12}^2 \Gamma_{21}^2 &= \Gamma_{21}^2 \Gamma_{11}^1 + \Gamma_{22}^2 \Gamma_{11}^2 \\ &\text{for the } \partial_2\text{-component of } [\nabla_1, \nabla_2] \partial_1. \end{aligned}$$

Subtracting (4.7) from (4.6), $[\nabla_1, \nabla_2] \partial_2 = 0$ implies:

$$(4.10) \quad \begin{aligned} \Gamma_{11}^1 \Gamma_{22}^1 + \Gamma_{12}^1 \Gamma_{22}^2 &= \Gamma_{21}^1 \Gamma_{12}^1 + \Gamma_{22}^1 \Gamma_{12}^2 \\ &\text{for the } \partial_1\text{-component of } [\nabla_1, \nabla_2] \partial_2. \end{aligned}$$

$$(4.11) \quad \begin{aligned} \Gamma_{11}^2 \Gamma_{22}^1 &= \Gamma_{21}^2 \Gamma_{12}^1 \\ &\text{for the } \partial_2\text{-component of } [\nabla_1, \nabla_2] \partial_2. \end{aligned}$$

Now apply (4.3) to rewrite (4.8), (4.9), (4.10), and (4.11) in terms of the four variables

$$\Gamma_{11}^1, \Gamma_{11}^2, \Gamma_{22}^1, \Gamma_{22}^2.$$

First, (4.8) becomes

$$(4.12) \quad \Gamma_{11}^1 \Gamma_{22}^2 = \Gamma_{11}^2 \Gamma_{22}^1,$$

Since ∇ is equiaffine, the curvature tensor takes values in traceless endomorphisms. Thus the ∂_1 -component of $[\nabla_1, \nabla_2] \partial_2$ is the negative of ∂_2 -component of $[\nabla_1, \nabla_2] \partial_1$. Thus (4.11) is equivalent to (4.8) and (4.11) provides nothing new.

Next, (4.9) and (4.10) respectively become:

$$(4.13) \quad (\Gamma_{11}^1)^2 = \Gamma_{11}^2 \Gamma_{22}^2$$

$$(4.14) \quad (\Gamma_{22}^2)^2 = \Gamma_{11}^1 \Gamma_{22}^1.$$

The three equations (4.12), (4.13) and (4.14) now describe the twisted cubic cone \mathfrak{C} as in §1, by taking:

$$\begin{aligned} U &= \Gamma_{11}^2 \\ X &= \Gamma_{11}^1 \\ Y &= \Gamma_{22}^2 \\ Z &= \Gamma_{22}^1, \end{aligned}$$

thus identifying the deformation space $\mathcal{CA}(\mathbb{T}^2)$ of marked complete affine structures on \mathbb{T}^2 with \mathfrak{C} .

Acknowledgments

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