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A 3-Manifold with no Real Projective Structure

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Dedicated to Michel Boileau on the occasion of his 60*th birthday*

Résumé. — Nous montrons que la somme connexe de deux copies de l'espace projectif de dimension trois n'admet pas de structure projective réelle. Ceci est le premier exemple connu d'une variété connexe de dimension 3 sans structure projective réelle.

Abstract. — We show that the connected sum of two copies of real projective 3-space does not admit a real projective structure. This is the first known example of a connected 3-manifold without a real projective structure.

Contents

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1. Introduction

Geometric structures modeled on homogeneous spaces of Lie groups were introduced by Ehresmann [14]. If X is a manifold upon which a Lie group G acts transitively, then an *Ehresmann structure* modeled on the homogeneous space (G, X) is defined by an atlas of coordinate charts into X such that the coordinate changes locally lie in G. For example, an Ehresmann structure modeled on Euclidean geometry is equivalent to a flat Riemannian metric. More generally, constant curvature Riemannian metrics are Ehresmann structures modeled on the sphere or hyperbolic space and their respective groups of isometries. A recent survey of the theory of Ehresmann structures on low-dimensional manifolds is [18]. Ehresmann (G, X)-structures are special cases of flat Cartan connections (modeled on (G, X)) with vanishing curvature. See Sharpe [26] for a modern treatment of this theory.

Topological uniformization in dimension 2 asserts that every closed 2-manifold admits a constant curvature Riemannian metric. Therefore every such surface is uniformized by one of three Ehresmann structures corresponding to constant curvature Riemannian geometry. However, projective and conformal geometry provide two *larger geometries,* each of which uniformize *all* surfaces (Ehresmann [14]).

The subject received renewed attention in the late 1970's by W. Thurston, who cast his Geometrization Conjecture (now proved by Perelman) in terms of Ehresmann (G, X)-structures. Thurston proposed that the relevant geometries are *locally homogeneous* 3*-dimensional Riemannian manifolds.* These are the 3-dimensional homogeneous spaces G/H where the isotropy group H is compact. Up to local isometry, those which cover compact 3-manifolds fall into eight types. See Scott [25], Thurston [29] and Bonahon [5] for a description of these geometries. Every closed 3-manifold canonically decomposes along essential elliptic or Euclidean 2-manifolds into pieces, each of which admit a geometric structure of one of these eight types.

Since these eight geometries often themselves admit geometric structures modeled on homogeneous spaces with *noncompact* isotropy group, it is tempting to search for geometries which uniformize *every* closed 3-manifold. [15] exhibits examples of closed 3-manifolds which admit no flat conformal structures. ([15] also contains examples of 3-manifolds, such as the 3-torus, which admit no spherical CR-structure.) The purpose of this note is to exhibit a closed 3-manifold (namely the connected sum $\mathbb{RP}^3 \# \mathbb{RP}^3$) which does not admit a flat *projective* structure. (On the other hand $\mathbb{RP}^3 \# \mathbb{RP}^3$ does admit a flat conformal and spherical CR structures.)

A $\mathbb{R}\mathsf{P}^n$*-structure* on a connected smooth n-manifold M is a Ehresmann structure modeled on $\mathbb{R}\mathsf{P}^n$ with coordinate changes locally in the group $\mathsf{PGL}(\mathsf{n}+1,\mathbb{R})$ of *collineations* (projective transformations) of $\mathbb{R}\mathsf{P}^n$. Such a structure is defined by an atlas for M where the transition maps are the restrictions of projective transformations to open subset of projective n-space. Fix a universal covering space $\tilde{M} \to M$; then an atlas as above determines an immersion called the *developing map*

$$\tilde{M} \xrightarrow{\mathsf{dev}} \mathbb{R}\mathsf{P}^n$$

and a homomorphism called the *holonomy:*

$$\pi_1 M \xrightarrow{\mathsf{hol}_M} \mathsf{PGL}(\mathsf{n}+1,\mathbb{R})$$

such that for all $\tilde{m} \in \tilde{M}$ and all $g \in \pi_1 M$ that

$$\mathsf{dev}_M(g \cdot \tilde{m}) \;=\; \mathsf{hol}_M(g) \cdot \mathsf{dev}_M(\tilde{m}).$$

Basic questions include the existence and classification of $\mathbb{R}\mathsf{P}^3$-structures on a given 3-manifold.

Recent progress on classification is documented in [12],[11]: in particular certain closed hyperbolic 3-manifolds admit continuous families of projective structures containing the hyperbolic structure, while others do not.

Every 2-manifold Σ admits a projective structure. The convex ones form a cell of dimension $16\mathsf{genus}(\Sigma)$ (Goldman [17]). Suhyoung Choi [8] showed that every $\mathbb{R}\mathsf{P}^2$-manifold of genus $g > 1$ decomposes naturally into convex subsurfaces. Combining these two results completely classify $\mathbb{R}\mathsf{P}^2$-structures [10],[9]. Almost all geometric 3-manifolds admit a projective structure, in fact:

THEOREM. — *Suppose that M is a 3-manifold equipped with one of the eight Thurston geometric structures. Then either M is a Seifert fiber space with a fibration that does not admit an orientation (and there is a double cover which is real projective) or else M inherits a uniquely determined real projective structure underlying the given Thurston geometric structure.*

All this was presumably known to Thurston, and was documented by Thiel[27] and Molnar [23]. This theorem is a consequence of the existence of a representation of each of the eight Thurston geometries (G, X) into $(\mathbb{R}\mathsf{P}^3, \mathsf{PGL}(4,\mathbb{R}))$ except that in the case of the product geometries $S^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$ the group $G = Isom(X)$ is replaced by the index-2 subgroup $\mathsf{Isom}_+(X)$, which preserves the orientation on the $\mathbb{R}$ direction. In general some 3-manifolds admit a real projective structure that is not obtained from a Thurston geometric structure (Benoist [3]). Furthermore exceptional fibered examples admit exotic real projective structures which do not arise from a projective representation of the associated geometry. (Compare Guichard-Wienhard [19] for some examples on twisted S^1-bundles over closed hyperbolic surfaces.)

The manifold $\mathbb{R}\mathsf{P}^3\#\mathbb{R}\mathsf{P}^3$ admits a geometric structure modeled on $S^2\times\mathbb{R}$. Our main result is:

Theorem. — *The 3-manifold* $M = \mathbb{R}\mathsf{P}^3\#\mathbb{R}\mathsf{P}^3$ *does not admit an* $\mathbb{R}\mathsf{P}^3$*-structure.*

One impetus to prove this result is the fact that almost all geometric 3-manifolds in the sense of Thurston have projective structures. This suggested that such structures might be universal for 3-manifolds, an outcome that would have had significant consequences since, for example, every closed simply connected projective manifold is a sphere. (This would imply the Poincaré conjecture.) One could imagine a functional, analogous to the Cherns-Simon invariant (whose critical points are conformally flat metrics) or the projective Weyl tensor, whose gradient flow would converge to a flat projective structure. Instead, as our simple example shows, the situation turns out to be more intriguing and complex.

After proving this result we learned from Yves Benoist that this result can also be deduced from his classification [1, 2] of real projective manifolds with abelian holonomy. However we believe that our direct proof, without using Benoist's general machinery, may suggest generalizations. A key point in Benoist's classification (see [2], §4.4 and Proposition 4.9 in particular) is that the developing image of such an $\mathbb{R}\mathsf{P}^3$-structure is the complement of a disjoint union of projective subspaces of dimension 0 (a point) and 2, presenting a basic asymmetry which is incompatible with the deck transformation of $\hat{M}$. This is impossible as described in the next paragraph. The example in §3 is a projective 3-manifold whose *holonomy* is infinite dihedral but not injective. In this case the developing image is the complement of two projective lines, which deformation retracts to a two-torus, and has trivial holonomy. In *some sense*[1] this manifold is trying to be $\mathbb{R}\mathsf{P}^3\#\mathbb{R}\mathsf{P}^3$.

To give some intuition for the following proof we first show that the developing map for a real projective structure on $\mathbb{R}\mathsf{P}^3\#\mathbb{R}\mathsf{P}^3$ cannot be injective. The universal cover of M is $S^2 \times \mathbb{R}$. If the developing map embeds this in $\mathbb{R}\mathsf{P}^3$ then there are two complementary components and they have

(1) A phrase the first author *learned* from Michel Boileau

the homotopy type of a point and $\mathbb{P}^2$. There is a covering transformation of the universal cover which swaps the ends. The holonomy leaves the image of the developing map invariant but swaps the complementary components. This is of course impossible since they have different homotopy types. Unfortunately one can't in general assume the developing map for a projective structure is injective.

Currently, it seems to be very difficult to show that a 3-manifold does not admit a projective structure. We do not know if a connected sum can ever admit a projective structure. Is there a projective structure on a closed Seifert-fibered manifold $\neq S^3$ for which the holonomy of the fiber is trivial? In this regard, we note that Carrière-d'Albo-Meignez [7] have shown that several closed Seifert 3-manifolds do not admit affine structures.

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2. The Ehresmann-Weil-Thurston principle

Fundamental in the deformation theory of locally homogeneous (Ehresmann) structures is the following principle, first observed by Thurston [28]:

Theorem 2.1. — *Let X be a manifold upon which a Lie group G acts transitively. Let M have a geometric structure modeled on (G,X) with holonomy representation $\pi_1(M) \xrightarrow{\rho} G$. For ρ' sufficiently near ρ in the space of representations $\mathsf{Hom}(\pi_1(M),G)$, there exists a (nearby) (G,X)-structure on M with holonomy representation ρ'.*

Corollary 2.2. — *Let M be a closed manifold. The set of holonomy representations of (G,X)-structures on M is open in $\mathsf{Hom}(\pi_1(M),G)$.*

This principle has a long history. In the context of $\mathbb{C}\mathsf{P}^1$-structures, this is due to Hejhal [20]; see also Earle [13] and Hubbard [21]. The first application is the theorem of Weil [31] that the set of Fuchsian representations of the fundamental group of a closed surface group in $\mathsf{PSL}(2,\mathbb{R})$ is open. The first detailed proofs of this fact are Lok [22], Canary-Epstein-Green [6], and Goldman [16] (the proof in [16] was worked out with M. Hirsch, and were independently found by A. Haefliger). The ideas in these proofs may be traced to Ehresmann. For a more recent proof, with applications to rigidity, see Bergeron-Gelander [4].

In the sequel $M = \mathbb{R}\mathsf{P}^3 \# \mathbb{R}\mathsf{P}^3$. By Van Kampen's theorem,

$$\pi_1 M \;\cong \langle\; a, b \;:\; a^2 = 1 = b^2 \;\rangle$$

is isomorphic to the infinite dihedral group.

3. An example with dihedral holonomy

Although we prove that no $\mathbb{R}\mathsf{P}^3$-structure exists on $\mathbb{R}\mathsf{P}^3 \# \mathbb{R}\mathsf{P}^3$, there do exist $\mathbb{R}\mathsf{P}^3$-manifolds whose holonomy is the infinite dihedral group. Namely, consider two linked projective lines ℓ_1, ℓ_2 in $\mathbb{R}\mathsf{P}^3$ and a collineation γ having ℓ_1 as a sink and ℓ_2 as a source. Then the complement

$$\Omega := \mathbb{R}\mathsf{P}^3 \setminus (\ell_1 \cup \ell_2)$$

is fibered by 2-tori and the region between two of them forms a fundamental domain for the cyclic group $\langle\gamma\rangle$ acting on Ω. The quotient $\Omega/\langle\gamma\rangle$ is an $\mathbb{R}\mathsf{P}^3$-manifold diffeomorphic to a 3-torus having cyclic holonomy group.

Now choose an free involution ι of $\mathbb{R}\mathsf{P}^3$ which interchanges ℓ_1 and ℓ_2, conjugating γ to γ^{-1}. The group $\Gamma := \langle\gamma, \iota\rangle$ acts properly and freely on Ω and contains the cyclic subgroup $\langle\gamma\rangle$ with index two. The quotient Ω/Γ is an $\mathbb{R}\mathsf{P}^3$-manifold with cyclic holonomy. It is a Bieberbach manifold, having a *Euclidean structure.*

In coordinates we may take ℓ_1 and ℓ_2 to be the projectivizations of the linear subspaces $\mathbb{R}^2 \times \{0\}$ and $\{0\} \times \mathbb{R}^2$ respectively. The projective transformations γ and ι are represented by the respective matrices:

$$\gamma \longleftrightarrow \begin{bmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda^{-1} & 0 \\ 0 & 0 & 0 & \lambda^{-1} \end{bmatrix}, \quad \iota \longleftrightarrow \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

4. Proof of Main Theorem

Using the presentation of $\pi_1 M$ above there is a short exact sequence

$$1 \longrightarrow \mathbb{Z} \longrightarrow \pi_1 M \;\cong\; \mathbb{Z}_2 * \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2 \longrightarrow 1.$$

and the product $c := ab$ generates the infinite cyclic normal subgroup. Corresponding to the subgroup of $\pi_1 M$ generated by a and c^n there is an

n-fold covering space $M^{(n)} \to M$. The manifold $M^{(n)}$ is homeomorphic to M. When $n = 2$ the cover is regular and corresponds to the subgroup generated by a and bab^{-1}. Thus any projective structure on M yields other projective structures on (covers of) M whose holonomy has certain desirable properties. We use this trick throughout the paper.

If M admits an $\mathbb{R}\mathsf{P}^3$ structure, then there is a developing map $\mathsf{dev}_M : \tilde{M} \to \mathbb{R}\mathsf{P}^3$ with holonomy $\mathsf{hol}_M : \pi_1(M) \to \mathsf{PGL}(4,\mathbb{R})$. Choose $A, B \in \mathsf{GL}(4,\mathbb{R})$ with $[A] = \mathsf{hol}_M(a),\ [B] = \mathsf{hol}_M(b)$. Set $C = AB$.

In view of the previous remarks, after passing to the double covering-space $M^{(2)}$, there is a projective structure with the matrices A and B conjugate. This property continues to hold after passing to a further n-fold covering space $M^{(2n)} \longrightarrow M^{(2)}$, thereby replacing C^2 by C^{2n}. This covering, combined with a small deformation, enables one to reduce the problem to a restricted class of holonomies.

Outline proof. If M admits a projective structure then after a small deformation some finite covering is $N = S^2 \times S^1$ with a projective structure with holonomy contained in a one-parameter group G that becomes diagonal after conjugacy. Furthermore there is an involution, τ, of N reversing the S^1 factor which is realized by a projective map which normalizes G. The flow generated by G on $\mathbb{R}\mathsf{P}^3$ pulls back to N. The flow on $\mathbb{R}\mathsf{P}^3$ has stationary points consisting of certain projective subspaces corresponding to the eigenspaces of G. One quickly reduces to the case that the flow on N is periodic giving a product structure. The orbit space is S^2. The orbit space of the flow on $\mathbb{R}\mathsf{P}^3$ is a non-Hausdorff surface $\mathcal{L}$. The developing map induces an immersion of S^2 into $\mathcal{L}$. There are only two possibilities for $\mathcal{L}$ corresponding to the two structures of the stationary set. The possibilities for immersions of S^2 into $\mathcal{L}$ are determined. None of these is compatible with the action of τ. This contradicts the existence of a developing map. $\square$

Lemma 4.1. — *The holonomy is injective.*

Proof. — Otherwise the holonomy has image a proper quotient of the infinite dihedral group which is therefore a finite group. The cover $\tilde{M}' \to M$ corresponding to the kernel of the holonomy is then a finite cover which is immersed into $\mathbb{R}\mathsf{P}^3$ by the developing map. Since $\tilde{M}'$ is compact dev is a covering map. Hence $\tilde{M}'$ is a covering-space of $\mathbb{R}\mathsf{P}^3$. But $\pi_1\tilde{M}'$ is infinite, which contradicts that it is isomorphic to a subgroup of $\pi_1\mathbb{R}\mathsf{P}^3 \cong \mathbb{Z}_2$. $\square$

Observe that in $\pi_1 M$ that c is conjugate to c^{-1} since

$$c^{-1} = (ab)^{-1} = b^{-1}a^{-1} = ba = b(ab)b^{-1} = bcb^{-1}.$$

It follows that for each eigenvalue λ of C the multiplicity of λ is the same as that of λ^{-1}.

Lemma 4.2. — *We may assume C is diagonalizable over $\mathbb{R}$ and has positive eigenvalues.*

Proof. — After passing to the double cover of M discussed above we may assume that A and B are conjugate. Since $[A]^2 \in \mathsf{PGL}(4,\mathbb{R})$ is the identity it follows that after rescaling A we have $A^2 = \pm Id$, thus A is diagonalizable over $\mathbb{C}$. If $A^2 = Id$ then A has eigenvalues ± 1. Since we are only interested in $[A]$ we may multiply A by -1 and arrange that the eigenvalue -1 has multiplicity at most 2. Otherwise $A^2 = -Id$ and A has eigenvalues eigenvalues $\pm i$ each with multiplicity two. Thus A is *conjugate* in $\mathsf{GL}(4,\mathbb{R})$ to one of the matrices:

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

After conjugating hol we may further assume that $A = A_i$ for some $i \in \{1,2,3\}$. Since A and B are conjugate there is $P \in \mathsf{GL}(4,\mathbb{R})$ such that $B = P \cdot A \cdot P^{-1}$. Then $C = A \cdot P \cdot A \cdot P^{-1}$. Changing P is a way to deform hol. The first step is to show that when P is in the complement of a certain algebraic subset then C has four distinct eigenvalues and is therefore diagonalizable over $\mathbb{C}$.

Given a homomorphism $\mathsf{hol}' : \pi_1 M \to \mathsf{PGL}(4,\mathbb{R})$ sufficiently close to hol by 2.1 there is a projective structure on M with this holonomy. Consider the map

$$f : \mathsf{GL}(4,\mathbb{R}) \longrightarrow \mathsf{SL}(4,\mathbb{R})$$

given by

$$f(P) = A \cdot P \cdot A \cdot P^{-1}.$$

This is a regular map defined on $\mathsf{GL}(4,\mathbb{R})$. Define $g : \mathsf{SL}(4,\mathbb{R}) \to \mathbb{R}^2$ by $g(Q) = (\mathsf{trace}(Q), \mathsf{trace}(Q^2))$. This is also a regular map.

Case 1. $A = A_1$ or A_3. An easy computation shows that the image of $g \circ f$ contains an open set:

A	P	$g \circ f$
A_1	$\begin{bmatrix} 0 & 1 & 0 & 0 \\ y & 0 & 0 & 0 \\ 1 & 0 & 0 & x \\ 0 & 0 & 1 & 1 \end{bmatrix}$	$x^{-2}y^{-2}(x^2y + 2xy^2 + x^3y^2 + x^2y^3, x^2 + 4y^2 + 2x^2y^2 + x^4y^2 + x^2y^4)$
A_3	$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ y+x & 0 & 0 & y-x \\ 0 & 1 & 0 & 0 \end{bmatrix}$	$x^{-2}(-2x^2 - 2xy,\ 4y^2)$

The subset $E \subset \mathsf{GL}(4, \mathbb{R})$ consisting of all P for which $C = f(P)$ has a repeated eigenvalue is the affine algebraic set where the discriminant of the characteristic polynomial of C vanishes.

Subclaim. — E is a proper subset.

Let S be the set of eigenvalues of C. The map $\tau(z) = z^{-1}$ is an involution on S because C is conjugate to C^{-1}. Each orbit in S under this involution contains at most 2 elements. An orbit of size one consists of either 1 or -1, from which it follows that if $P \in E$ then $|S| < 4$ and $S \subset \{\pm 1, \lambda^{\pm 1}\}$. Thus if $P \in E$ either $S \subset \{\pm 1\}$ or

$$\mathsf{trace}(C) = \lambda + \lambda^{-1} + m \quad \textit{or} \quad \mathsf{trace}(C) = 2\lambda + 2\lambda^{-1}$$

where $m \in \{0, \pm 2\}$ and

$$\mathsf{trace}(C^2) = \lambda^2 + \lambda^{-2} + 2.$$

In each case $\mathsf{trace}(C)$ and $\mathsf{trace}(C^2)$ satisfies an algebraic relation. Thus $\dim[g \circ f(E)] = 1$. The image of $g \circ f$ contains an open set therefore E is a proper subset, proving the subclaim. $\square$

Since E is an algebraic subset of $\mathsf{GL}(4, \mathbb{R})$ which is a proper subset it follows that $\mathsf{GL}(4, \mathbb{R}) \setminus E$ is open and dense in the Euclidean topology. Hence there is a small perturbation of P and of hol so that C is diagonalizable over $\mathbb{C}$ and has 4 distinct eigenvalues $\{\lambda_1^{\pm 1}, \lambda_2^{\pm 1}\}$.

By suitable choice of P, we can arrange that the arguments of λ_1 and λ_2 are rational multiples of π. Furthermore passing to a finite covering-space of M, we may assume all eigenvalues of C are real. Passing to a double covering-space we may assume these eigenvalues are positive. However it is possible that they are no longer distinct.

We have shown in this case that the projective structure on (a finite cover of) M may be chosen so that C is diagonal with real positive eigenvalues, which completes **case 1**. $\square$

Case 2. $A = A_2$. Then for every choice of P the $+1$ eigenspaces of A and B intersect in a subspace of dimension at least 2. Since $C = AB$ it follows that there is a 2-dimensional subspace on which C is the identity, and thus C has eigenvalue 1 with multiplicity at least 2. It is easy to see that $\mathsf{trace} \circ f$ is not constant, for example when

$$P = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & x & 0 & 1 \end{bmatrix} \qquad \mathsf{trace}(f(P)) = 4/(1+x)$$

Thus on a dense open set $f(P) \neq 4$ so C has an eigenvalue $\lambda \neq 1$. As before, by replacing C by C^2 if needed, we may assume $\lambda \neq \pm 1$. Thus $\lambda^{-1} \neq \lambda$ is also an eigenvalue giving 3 distinct eigenvalues $\lambda, \lambda^{-1}, 1, 1$. Since the $+1$-eigenspace of C has dimension two, C is diagonalizable over $\mathbb{C}$. The rest of the argument is as before. $\square$

Lemma 4.3. — *We may assume that C is one of the following matrices with $\lambda_2 > \lambda_1 > 1$.*

$$C_1 = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_1^{-1} & 0 \\ 0 & 0 & 0 & \lambda_1^{-1} \end{bmatrix}, \quad C_2 = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_1^{-1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$C_3 = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_1^{-1} & 0 \\ 0 & 0 & 0 & \lambda_2^{-1} \end{bmatrix}$$

Proof. — The result follows from Lemma 4.2 and the fact that C is conjugate to C^{-1}. Observe that Lemma 4.1 rules out all eigenvalues are 1. $\square$

There is a 1-parameter diagonal subgroup $g : \mathbb{R} \to G \subset \mathsf{PGL}(4, \mathbb{R})$ such that $g(1) = [C]$. For example if $C = C_3$ then this subgroup is:

$$g_1(t) = \begin{bmatrix} \exp(\ell_1 t) & 0 & 0 & 0 \\ 0 & \exp(\ell_2 t) & 0 & 0 \\ 0 & 0 & \exp(-\ell_1 t) & 0 \\ 0 & 0 & 0 & \exp(-\ell_2 t) \end{bmatrix} \qquad \ell_i = \log(\lambda_i).$$

This group G is characterized as the unique one-parameter subgroup which contains the cyclic group H generated by C and such that every element in G has real eigenvalues. Since H is normal in $\mathsf{hol}(\pi_1 M)$ it follows from the characterization that G is normalized by $\mathsf{hol}(\pi_1 M)$.

Let $N \to M$ be the double cover corresponding to the subgroup of $\pi_1 M$ generated by c. Observe that $N \cong S^2 \times S^1$. Let $\pi : \tilde{N} \to N$ be the universal cover. Then N inherits a projective structure from M with the same developing map $\mathsf{dev}_N = \mathsf{dev}_M$. The image of the holonomy for this projective structure on N is generated by $[C]$. Let $z \in gl(4, \mathbb{R})$ be an infinitesimal generator of G so that $G = \exp(\mathbb{R} \cdot z)$. Thus for C_3 we have

$$z = \begin{bmatrix} \ell_1 & 0 & 0 & 0 \\ 0 & \ell_2 & 0 & 0 \\ 0 & 0 & -\ell_1 & 0 \\ 0 & 0 & 0 & -\ell_2 \end{bmatrix}.$$

There is a flow $\Phi : \mathbb{R}\mathsf{P}^3 \times \mathbb{R} \to \mathbb{R}\mathsf{P}^3$ on $\mathbb{R}\mathsf{P}^3$ generated by G given by

$$\Phi(x, t) = \exp(tz) \cdot x.$$

Let V be the vector field on $\mathbb{R}\mathsf{P}^3$ velocity of this flow. The fixed points of the flow are the zeroes of this vector field. The vector field is preserved by the flow, and thus by $\mathsf{hol}(\pi_1 N)$. It follows that V pulls back via the developing map to a vector field $\tilde{v}$ on $\tilde{N}$ which is invariant under covering transformations and thus covers a vector field v on N.

The subset $Z \subset \mathbb{R}\mathsf{P}^3$ on which V is zero is the union of the eigenspaces of C. Thus the possibilities for the zero set Z are:
(1) For C_1 two disjoint projective lines.
(2) For C_2 one projective line and two points.
(3) For C_3 four points.

Lemma 4.4. — *$C = C_1$ is impossible.*

Proof. — If $C = C_1$ then Z is the union of disjoint two lines ℓ_1, ℓ_2 in $\mathbb{R}\mathsf{P}^3$ which are invariant under $\mathsf{hol}(\pi_1 N)$. Then $\mathsf{dev}^{-1}(\ell_i)$ is a 1-submanifold in $\tilde{N}$ which is a closed subset invariant under covering transformations. Hence

$$\alpha_i = \pi(\mathsf{dev}^{-1}(\ell_i))$$

is a compact 1-submanifold in N. Furthermore $\alpha_1 \cup \alpha_2$ is the zero set of v. We claim $\alpha_1 \cup \alpha_2$ is not empty; equivalently v must be zero somewhere in

N. Otherwise
$$\mathsf{dev} : \tilde{N} \to X \equiv \mathbb{R}\mathsf{P}^3 \setminus (\ell_1 \cup \ell_2)$$
covers an immersion
$$N \to X/\mathsf{hol}(\pi_1(N)) \cong T^3.$$

This is an immersion of one closed manifold into another of the same dimension and is thus a covering map. However $N \cong S^1 \times S^2$ is not a covering space of T^3 since the latter has universal cover Euclidean space and the former has universal cover $S^2 \times \mathbb{R}$.

Thus we may suppose α_1 is not empty. Let β be the closure of a flowline of v with one endpoint on α_1. Now β is a compact 1-submanifold of N because its pre-image in $\tilde{N}$ develops into a closed invariant interval in $\mathbb{R}\mathsf{P}^3$ with one endpoint in each of ℓ_1 and ℓ_2. Thus β has the other endpoint in α_2 which is therefore also non-empty. We claim that α_1 is connected and isotopic in $N = S^2 \times S^1$ to $* \times S^1$. But this is impossible, for $A = S^2 \times *$ intersects α_1 once transversely. But A lifts to $\tilde{A} \subset \tilde{N}$ and then $\mathsf{dev}(\tilde{A})$ is an immersion of a sphere into $\mathbb{R}\mathsf{P}^3$ which meets $\ell_1 = \mathsf{dev}(\pi^{-1}\alpha_1)$ once transversely. However
$$[\ell_1] = 0 \in H_1(\mathbb{R}\mathsf{P}^3, \mathbb{Q})$$
and intersection number is an invariant of homology classes, so this is impossible.

It remains to show α_1 is connected and isotopic to $S^1 \times *$. Let γ_1 be a component of α_1. Let U be the basin of attraction in N of γ_1. Now $\mathsf{dev}(\pi^{-1}\gamma_1) \subset \ell_1$ and an easy argument shows these sets are equal. Hence $\mathsf{dev}(\pi^{-1}(U))$ contains a neighborhood of ℓ_1. Thus U contains a small torus transverse to the flow and bounding a small neighborhood of γ_1. Since U is preserved by the flow if follows that $U \cong T^2 \times \mathbb{R}$. The frontier of U in N is contained in $\alpha_1 \cup \alpha_2$. Hence α_1, α_2 are both connected and $N = \alpha_1 \cup U \cup \alpha_2$. Thus $N = H_1 \cup H_2$ where
$$H_i = \alpha_i \cup T^2 \times (0, 1] \cong S^1 \times D^2.$$

This gives a genus-1 Heegaard splitting of $N = S^2 \times S^1$. By Waldhausen [30] such a splitting is standard. In particular this implies that $\alpha_1 = \gamma_1$ is isotopic to $S^1 \times *$. □

We are reduced to the case that C is C_2 or C_3. In each case there is a unique isolated zero of V which is a source and another which is a sink.

Lemma 4.5. — $\mathsf{dev}(\tilde{N})$ *contains no source or sink.*

Proof. — By reversing the flow we may change a source into a sink. So suppose p is a sink in the image of the developing map. Let Q be the projective plane which contains the other points corresponding to the other eigenspaces of C. Then Q is preserved by $\mathsf{hol}(\pi_1 N)$. There is a decomposition into disjoint subspaces $\mathbb{R}\mathsf{P}^3 = p \cup \Omega \cup Q$ where $\Omega \cong S^2 \times \mathbb{R}$ is the basin of attraction for p. Furthermore each of these subspaces is invariant under $\mathsf{hol}(\pi_1 N)$. Thus there is a corresponding decomposition of N into disjoint subsets: $\pi(\mathsf{dev}^{-1}(p))$ is a finite non-empty set of points, $\pi(\mathsf{dev}^{-1}(Q))$ is a compact surface, and $\pi(\mathsf{dev}^{-1}(\Omega))$ an open submanifold.

Now Ω admits a foliation by concentric spheres centered on p which is preserved by the flow induced by V and hence by $\mathsf{hol}(\pi_1 N)$. This gives a foliation of $\mathbb{R}\mathsf{P}^3 \setminus p$ by leaves, one of which is $Q \cong \mathbb{P}^2$ and the others are spheres. Hence this induces a foliation of $N \setminus \pi(\mathsf{dev}^{-1}(p))$. Since $\pi(\mathsf{dev}^{-1}(p))$ is not empty every leaf near it is a small sphere. Thus N has a singular foliation where the singular points are isolated and have a neighborhood foliated by concentric spheres. It follows from the Reeb stability theorem [24] that if a compact connected 3-manifold has a foliation such that each component of the boundary is a leaf and some leaf is a sphere, then the manifold is $S^2 \times I$ or a punctured $\mathbb{R}\mathsf{P}^3$. But this contradicts that the manifold is $S^2 \times S^1$ minus some open balls. □

Lemma 4.6. — *The flow on N given by v is periodic and the flow lines fiber N as a product $S^2 \times S^1$.*

Proof. — Let λ be the closure of a flowline of V in $\mathbb{R}\mathsf{P}^3$ which has endpoints on the source and sink of V. Such flowlines are dense therefore we may choose λ to contain a point in $dev(\tilde{N})$. Then $dev^{-1}(\lambda)$ is a non-empty closed subset of $\tilde{N}$ which is a 1-submanifold without boundary, since the source and sink are not in $\mathsf{dev}(\tilde{N})$. Hence $\pi(\mathsf{dev}^{-1}\lambda)$ is a compact non-empty 1-submanifold in N. Let γ be a component. If γ were contractible in N then it would lift to a circle in $\tilde{N}$ and be mapped by the developing map into λ. But this gives an immersion of a circle into a line which is impossible. Thus $[\gamma] \neq 0 \in \pi_1(N)$.

Let $T > 0$ be the period of the closed flow line γ. Let U be the subset of N which is the union of closed flow lines of period T. We will show U is both open and closed. Since U is not empty and N is connected, the claim follows.

Choose a small disc, $D \subset N$, transverse to the flow and meeting γ once. Let $\tilde{D} \subset \tilde{N}$ be a lift which meets the component $\tilde{\gamma} \subset \pi^{-1}(\gamma)$. The union, $\tilde{Y}$, of the flowlines in $\tilde{N}$ which meets $\tilde{D}$ maps homeomorphically by the developing map into a foliated neighborhood of the interior of λ. Let τ be the covering transformation of $\tilde{N}$ given by $[\gamma] \in \pi_1(N)$. Then τ preserves

$dev(\tilde{Y})$ and preserves $\tilde{\gamma}$ therefore preserves $\tilde{Y}$. Furthermore

$$Y \;=\; \tilde{Y}/\tau \cong dev(\tilde{Y})/hol(\gamma) \;\cong\; S^1 \times D^2$$

is foliated as a product. Thus Y is a solid torus neighborhood of γ in N foliated as a product by flowlines. This proves U is open. The limit of flowlines of period T is a closed flowline with period T/n for some integer $n > 0$. But $n = 1$ since the set of flowlines of period T/n is open. Thus U is closed. □

Let $X = \mathbb{R}\mathsf{P}^3 \setminus Z$ be the subset where $V \neq 0$. Then X is foliated by flow lines. Let $\mathcal{L}$ be the leaf space of the foliation of X. Then $\mathcal{L}$ is a connected 2-manifold which may be non-Hausdorff. Since G is normalized by $\mathsf{hol}(\pi_1 M)$ it follows that this group acts on $\mathcal{L}$. Since $\mathsf{hol}(\pi_1 N) \subset G$ the action of $\mathsf{hol}(\pi_1 N)$ on $\mathcal{L}$ is trivial so the action of $\mathsf{hol}(\pi_1 M)$ on $\mathcal{L}$ factors through an action of $\mathbb{Z}_2$. Thus the holonomy gives an involution on $\mathcal{L}$. Below we calculate $\mathcal{L}$ and this involution in the remaining cases C_2, C_3.

Since $\mathsf{dev}(\tilde{N}) \subset X$ there is a map of the leaf space of the induced foliation on $\tilde{N}$ into $\mathcal{L}$. By Lemma 4.6 the leaf space of $\tilde{N}$ is the Hausdorff sphere S^2. The induced map $h : S^2 \to \mathcal{L}$ is a local homeomorphism, which we shall call an *immersion.* Since $\mathsf{dev}(\tilde{N}) \subset \mathbb{R}\mathsf{P}^3$ is invariant under $\mathsf{hol}(\pi_1 M)$ it follows that $h(S^2) \subset \mathcal{L}$ is invariant under the involution. Below we determine all immersions of S^2 into $\mathcal{L}$ and show that the image is never invariant under the involution. This means the remaining cases $C = C_2$ or $C = C_3$ are impossible, proving the theorem.

Lemma 4.7. — *Case $C = C_2$ is impossible.*

Proof. — The zero set of V consists of a point source, a point sink, and a $\mathbb{P}^1$ with hyperbolic dynamics in the transverse direction. Every flowline either starts at the source, or ends at the sink, or does both. Let S_1, S_2 be small spheres around the source and sink transverse to the flow. The quotient map $X \to \mathcal{L}$ embeds each of these spheres, and the union is all of $\mathcal{L}$.

It is easy to check that $\mathcal{L}$ is obtained from S_1 and S_2 by the following identifications. Regard each sphere as a copy of the unit sphere, S^2, in $\mathbb{R}^3$. Decompose this sphere into an equator and northern and southern hemispheres:

$$S^2 = D_+ \cup E \cup D_-$$

where

$$\begin{aligned} E &= S^2 \cap \{\ x_3 = 0\ \} \\ D_+ &= S^2 \cap \{\ x_3 > 0\ \} \\ D_- &= S^2 \cap \{\ x_3 < 0\ \}. \end{aligned}$$

Using the identifications of S_1 and S_2 with S^2 identify $D_+ \subset S_1$ with $D_+ \subset S_2$ using the identity map. Identify $D_- \subset S_1$ with $D_- \subset S_2$ using the map $(x_1, x_2, x_3) \mapsto (-x_2, x_1, x_3)$.

Thus $\mathcal{L}$ may be regarded as a sphere with with an extra copy of the equator. However one also needs to know a neighborhood basis for the points on the extra equator. This is determined by the above description. We show below that every immersed sphere in $\mathcal{L}$ is one of these two embedded spheres. The involution swaps S_1 and S_2 and therefore swaps the two equators in $\mathcal{L}$. The embedded spheres each contain only one equator and therefore there is no immersion of a sphere into $\mathcal{L}$ whose image is preserved by the involution.

It remains to determine the possible immersed spheres in $\mathcal{L}$. There is a decomposition of $\mathcal{L}$ into disjoint subsets, two of which are the points $(0, 0, \pm 1) \subset D_\pm$ and the other subsets are circles which foliate the complement. In particular each of the two equators is a leaf of this foliation.

Suppose A is a sphere and $h : A \to \mathcal{L}$ is an immersion. Then the pre-images of the decomposition give a decomposition of A. There are finitely many decomposition elements which are points. Call the set of these points P. Since h is an immersion, $A \setminus P$ is decomposed as a 1-dimensional foliation. Furthermore since A is compact and the 1-dimensional leaves in $\mathcal{L}$ are closed, their pre-images in A are compact thus circles. Thus $A \setminus P$ is foliated by circles and thus an open annulus. Hence the quotient space of A corresponding to the decomposition is a closed interval $\mathcal{I} \cong [-1, 1]$. The endpoints correspond to center type singularities of a singular foliation on A. The quotient space of the decomposition of $\mathcal{L}$ is a non-Hausdorff interval, $\mathcal{I}^* \cong [-1, 1] \cup \{0'\}$, with 2 copies of the origin. The endpoints correspond to the two decomposition elements that are points. The immersion h induces a map $\overline{h} : \mathcal{I} \to \mathcal{I}^*$. Since h is an immersion $\overline{h}$ is also an immersion (local homeomorphism). Thus $\overline{h}(\pm 1) = \pm 1$. The only such immersion is an embedding which contains one copy of the origin. This implies h is an embedding of the form claimed. □

It follows from the preceding results that $\mathsf{dev}(\tilde{N})$ is disjoint from the zeroes of the vector field.

LEMMA 4.8. — *Case $C = C_3$ is impossible.*

Proof. — The zero set of V consists of a 4 points. We label them as p_{+++}, p_{++-}, p_{+--}, p_{---}. The labelling reflects how many attracting and how many repelling directions there are. The number of $-$ signs is the number of attracting directions. Thus p_{---} is the sink, p_{+++} is the source.

Every flowline starts at a point with a $+$ label and ends at a point with a $-$ label. Every $\mathbb{P}^2$ containing three of these four points is invariant under the flow.

Let ℓ_- be the $\mathbb{P}^1$ containing p_{---} and p_{+--}. Let ℓ_+ be the $\mathbb{P}^1$ which contains p_{+++} and p_{++-}. The restriction of V to each of $\ell_\pm$ has on source and one sink and no other zeroes. There are thus two flowlines contained in each of $\ell_\pm$.

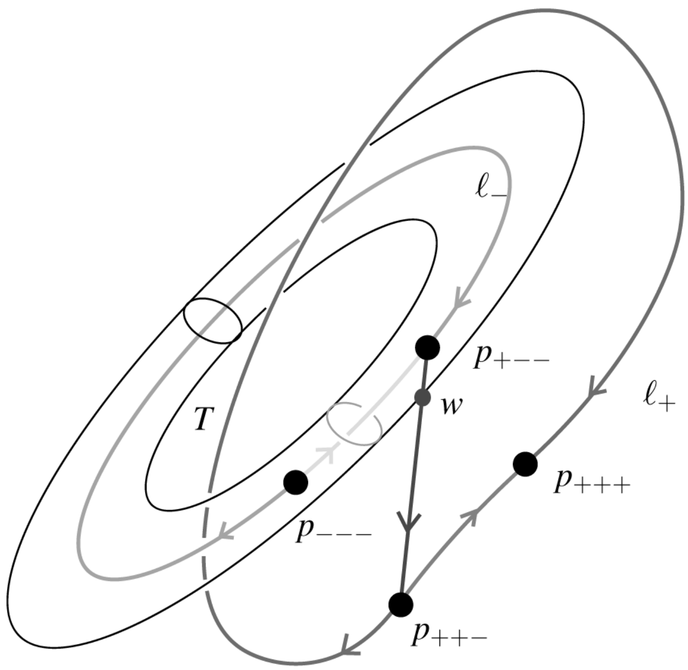

Figure 1. — Flowlines for case C_3.

Let T be a torus transverse to V and which is the boundary of a small neighborhood of ℓ_-. Then T intersects every flowline once except the 4 flowlines in $\ell_\pm$. Hence $\mathcal{L}$ may be identified with T plus 4 more points. Two of these points come from ℓ_+ and the other two from ℓ_-.

Since $aca^{-1} = c^{-1}$ it follows that $\mathsf{hol}(a)$ conjugates $\mathsf{hol}(c)$ to $\mathsf{hol}(c^{-1})$ and thus $\mathsf{hol}(a)$ permutes the zeroes of V by reversing the sign labels. Thus $p_{---} \leftrightarrow p_{+++}$ and $p_{+--} \leftrightarrow p_{++-}$.

Observe that T can be moved by the flow to a small torus around ℓ_+. Thus the involution on $\mathcal{L}$ maps the subset corresponding to T into itself and swaps the pair of points corresponding to ℓ_+ with the pair corresponding to ℓ_-. We show below that every immersion of a sphere into $\mathcal{L}$ contains either the pair of points corresponding to ℓ_+ or the pair corresponding to ℓ_- but not both pairs. As before the image of the developing map gives an immersion of a sphere into $\mathcal{L}$ which is preserved by the involution. Thus no such immersion exists and the remaining case $C = C_3$ is impossible.

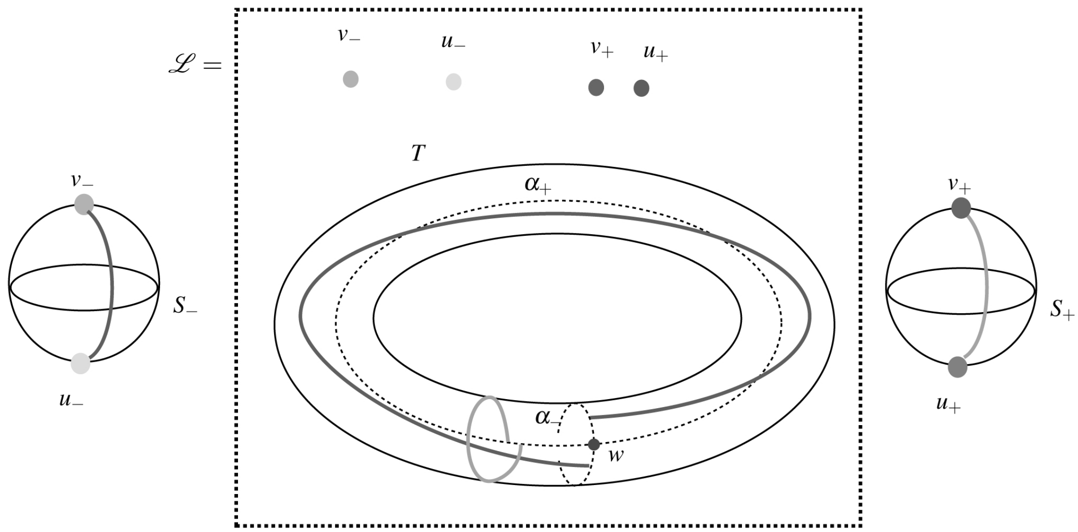

Figure 2. — Non-Hausdorff surface $\mathcal{L}$ for case C_3

We first describe $\mathcal{L}$ in a bit more detail. Let S_+ (resp. S_-) be a small sphere around p_{+++} (resp. p_{---}) transverse to the flow. Then every flowline meets $T \cup S_- \cup S_+$. We next describe the intersection of the images of T and $S_\pm$ in $\mathcal{L}$. We may choose S_- to be a small sphere inside T. The two flowlines in ℓ_- meet S_- but do not meet T. We call these points u_-, v_- in S_- and the corresponding points in $\mathcal{L}$ *exceptional points.* The remaining flowlines that meet S_- intersect T in the complement of the circle $\alpha_- \subset T$ where $\alpha_- = T \cap A_-$ and A_- is the $\mathbb{P}^2$ containing the four zeroes of V except p_{---}. A small deleted neighborhood in $\mathcal{L}$ of an exceptional point corresponding to a flowline in ℓ_- is an annulus on one side of α_-, either $\alpha_- \times (0,1)$ or $\alpha_- \times (-1,0)$, depending on which of the two exceptional points corresponding to a flowline in ℓ_- is chosen. Similarly the image of S_+ intersects the image of T in the complement of the circle $\alpha_+ = T \cap A_+$ where A_+ is the $\mathbb{P}^2$ containing the four zeroes of V except p_{+++}. The circles α_- and α_+ on T meet transversely at a single point w corresponding to the flowline between p_{+--} and p_{++-}.

Decompose $\mathcal{L}$ into subsets as follows. Decompose the image of T by circles given by a foliation of T by circles parallel to α_- and that are transverse to α_+. The remaining 4 exceptional points in $\mathcal{L}$ are also decomposition elements. Let A be a sphere and $h : A \to \mathcal{L}$ an immersion. As before we deduce that there is a finite set $P \subset A$ of decomposition elements which are points. The remaining decomposition elements give a foliation of $A \setminus P$. There is a small deleted neighborhood $U \subset A \setminus p$ of $p \in P$ such that $h(U)$ is an open annulus $\beta \times (0,1) \subset T$ whose closure consists of two disjoint circles either parallel to α_- or to $\alpha+_.$ It follows that the foliation on the subsurface $A_- \subset A$ with these small open neighborhoods of P removed has the property that each component of ∂A_- is either transverse to the foliation or is a leaf of the foliation. By doubling A_- along the boundary one obtains a foliation on a closed surface. Hence A_- is an annulus and the behavior of the foliation on both components of ∂A_- is the same. If the boundary components are leaves then $h(A)$ contains the two points corresponding to ℓ_-. Otherwise $h(A)$ contains the two points corresponding to ℓ_+. This completes the proof of the final case, and thus of the theorem.

We remark that the above discussion is similar to the case the developing map is injective discussed before the proof. We argued above that there is $S^2 \subset \tilde{N} \cong S^2 \times \mathbb{R}$ immersed in $\mathbb{P}^3$ by the developing map and with the source on the inside (relative to the flow) and the other three critical points on the outside. These three critical points lie on an $\mathbb{P}^2$ which is preserved by the flow. Indeed they are the critical points of a Morse function on this $\mathbb{P}^2$ given by the flow. In *some sense* the proof says this $\mathbb{P}^2$ is *outside* the immersed sphere. $\square$

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