THE LEVI-CIVITA CONNECTION FOR THE POINCARÉ METRIC

We denote complex numbers $z = x + yi \in \mathbb{C}$ where $x, y \in \mathbb{R}$.

Let H^2 denote the upper half-plane

$$\{x + iy | x, y \in \mathbb{R},\ y > 0\}$$

with the *Poincaré metric:*

$$g = \frac{|dz|^2}{y^2}$$

We compute the Levi-Civita connection ∇ with respect to several different frames.

1. The usual coordinate system on H^2

Let ∂_x, ∂_y be the coordinate vector fields, so that:

$$g(\partial_x, \partial_x) = g(\partial_y, \partial_y) = y^{-1} \tag{1}$$

$$g(\partial_x, \partial_y) = g(\partial_y, \partial_x) = 0. \tag{2}$$

The vector fields

$$\xi := y^{-1}\partial_x$$
$$\eta := y^{-1}\partial_y$$

define an orthonormal frame field.

Theorem. *In terms of the coordinate frame, the Levi-Civita connection is given by:*

$$\nabla_x \partial_x = y^{-1}\partial_y \tag{3}$$

$$\nabla_x \partial_y = -y^{-1}\partial_x \tag{4}$$

$$\nabla_y \partial_x = -y^{-1}\partial_x \tag{5}$$

$$\nabla_y \partial_y = -y^{-1}\partial_y \tag{6}$$

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In terms of the orthonormal frame, the Levi-Civita connection is given by:

$$\begin{aligned}\nabla_\xi \xi &= \eta\\ \nabla_\xi \eta &= -\xi\\ \nabla_\eta \xi &= 0\\ \nabla_\eta \eta &= 0\end{aligned}$$

Proof. Orthonormality implies $g(y\partial_x, y\partial_x) = 1$ is constant, whence

$$\begin{aligned}0 &= \partial_y g(y\partial_x, y\partial_x)\\ &= 2g(\nabla_y(y\partial_x), y\partial_x)\\ &= 2yg(\nabla_y(y\partial_x), \partial_x)\\ &= 2y(g(\partial_x, \partial_x) + yg(\nabla_y\partial_x, \partial_x))\\ &= 2y(y^{-1} + yg(\nabla_y\partial_x), \partial_x)\end{aligned}$$

and

$$g(\nabla_y\partial_x, \partial_x) = -y^{-3}. \tag{7}$$

Differentiating the constant $g(y\partial_y, \partial_y)$ with respect to x:

$$0 = 2\partial_x g(\nabla_x\partial_y, \partial_x) \tag{8}$$

whence

$$g(\nabla_y\partial_x, \partial_x) = 0. \tag{9}$$

Since ∇ is symmetric,

$$\nabla_x\partial_y = \nabla_y\partial_x. \tag{10}$$

Combining (9) with (10):

$$g(\nabla_x\partial_y, \partial_y) = -y^{-3}$$

and combining with (8), yields (4). Applying (10) again yields (5).

Differentiating $g(y\partial_x, y\partial_x) = 1$ with respect to ∂_x yields:

$$g(\nabla_x\partial_x, \partial_x) = 0 \tag{11}$$

Similarly, differentiating with respect to ∂_y;

$$g(\nabla_x\partial_x, \partial_x) = y^{-3} \tag{12}$$

which implies (3).

Differentiating $g(y\partial_x, y\partial_y) = 0$ with respect to ∂_y yields:

$$g(\nabla_y\partial_y, \partial_x) = 0 \tag{13}$$

Similarly, $0 = \partial_y g(\partial_y, \partial_y)$ implies;

$$g(\nabla_y\partial_y, \partial_y) = -y^{-3} \tag{14}$$

which implies (6).

The routine calculations for the orthonormal frame are omitted. □

2. Connection 1-form for orthonormal frame

Denote the coframe field dual to the orthonormal frame by:

$$\begin{aligned} \xi^* &:= y dx \\ \eta^* &:= y dy. \end{aligned}$$

The covariant differentials of the orthonormal frame are:

$$\begin{aligned} \nabla\xi &= \ \eta \ \ \xi^* \\ \nabla\eta &= \ -\xi \ \xi^* \end{aligned}$$

so the connection form is:

$$\begin{bmatrix} 0 & \xi^* \\ -\xi^* & 0 \end{bmatrix} \ = \ -y^{-1}dx \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

3. Geodesic curvature of a hypercycle

We use these to calculate the geodesic curvature of a hypercycle. The positive imaginary axis $i\mathbb{R}_+$ is a geodesic with endpoints $0, \infty$. The Euclidean rays from 0 subtending an angle θ with $i\mathbb{R}_+$ are hypercycles

$$\begin{aligned} re^{i\theta} &= r\big(\cos(\theta) + i\sin(\theta)\big) \\ &= r\big(\tanh(\rho) + i\,\mathrm{sech}(\rho)\big) \end{aligned}$$

at distance ρ from $i\mathbb{R}_+$. The point $x_0 + i$ has coordinates

$$x_0 = \frac{\tanh(\rho)}{\mathrm{sech}(\rho)} = \sinh(\rho).$$

The curve

$$\tilde{\gamma}(t) \ := \ e^t(x_0 + i)$$

has velocity vector and speed, respectively:

$$\begin{aligned} \tilde{\gamma}'(t) \ &:= \ e^t(x_0\partial_x + \partial_y) \\ &= x_0\xi + \eta \end{aligned}$$

$$\|\tilde{\gamma}'(t)\| = \sqrt{1 + x_0^2} \tag{15}$$

We compute the accelaration of $\tilde{\gamma}(t)$:

$$\begin{aligned} \frac{D}{dt}\tilde{\gamma}(t) &= \nabla_{x_0\xi+\eta}(x_0\xi + \eta) \\ &= x_0^2\eta - x_0\xi \end{aligned}$$

Now reparametrize $\tilde{\gamma}(t)$ by unit speed:

$$\gamma(t) := \frac{1}{\sqrt{1+x_0}} e^t (x_0 + i)$$

and the geodesic curve is the tangential component of the acceleration:

$$\frac{D}{dt}\gamma(t) = \frac{1}{1+x_0}\left(x_0^2\eta - x_0\xi\right)$$

which, since $\gamma(t)$ has constant speed, equals:

$$\begin{aligned}\left\|\frac{D}{dt}\gamma(t)\right\| &= \frac{|x_0|}{\sqrt{1+x_0^2}} \\ &= \sin(\theta) \\ &= \tanh(\rho)\end{aligned}$$

Similarly the geodesic curvature of a metric circle of radius ρ equals $\coth(\rho)$. To see this, use the Poincare unit disc model: for $|z| < 1$, the metric tensor is:

$$g := \frac{4|dz|^2}{(1-|z|^2)^2}$$

and writing (hyperbolic polar coordinates)

$$z = e^{i\theta}\tanh(\rho/2)$$

the metric tensor is $g = d\rho^2 + \sinh^2(\rho)d\theta^2$, with area form $dA = \sinh^2(\rho)d\rho \wedge d\theta$. Consider a disc D_ρ with (hyperbolic) radius ρ. Then its circumference equals $2\pi\sinh(\rho)$ and its area $2\pi(\cosh(\rho) - 1$. Let k_g be the geodesic curvature of the metric circle ∂D_ρ. Applying the Gauss-Bonnet theorem

$$2\pi\chi(\Sigma) = \int_\Sigma K dA + \oint_{\partial\Sigma} k_g ds$$

to $\sigma = D_\rho$ obtaining:

$$2\pi = -2\pi\big(\cosh(\rho) - 1) + k_g\big(2\pi\sinh(\rho)\big),$$

that is,

$$k_g = \coth(\rho)$$

as desired.

4. Fermi coordinates around a geodesic

Another convenient coordinate system begins with a geodesic and considers the family of geodesics orthogonal to this one. Let u denote the parameter along the geodesic, and v the parameter along the perpendiculars:

$$\begin{aligned} x &= e^u \tanh(v) \\ y &= e^u \operatorname{sech}(v). \end{aligned}$$

Then $\sinh(v) = x/y$ and $|z| = x^2 + y^2 = e^u$ and $u = \frac{1}{2}\log(x^2 + y^2)$. Writing

$$\begin{aligned} z &= e^u(\tanh(v) + i \operatorname{sech}(v)) \\ dz &= z(du + i \operatorname{sech}(v) dv) \\ |dz|^2 &= |z|^2\big(du^2 + \operatorname{sech}^2(v) dv^2\big) \end{aligned}$$

with metric tensor

$$g = \frac{|dz|^2}{y^2} = \cosh^2(v) du^2 + dv^2.$$

The coordinate 1-forms are:

$$\begin{aligned} du &= (x^2 + y^2)^{-1}(x\ dx + y\ dy) \\ dv &= (x^2 + y^2)^{-1/2}(y\ dx - x/y\ dy) \end{aligned}$$

and dual coodinate vector fields:

$$\begin{aligned} \partial_u &= \ x\ \partial_x + y\ \partial_y \\ \partial_v &= \ y(x^2 + y^2)^{-1/2}(y\ \partial_x - x\ \partial_y). \end{aligned}$$

5. The Levi-Civita connection in Fermi coordinates

Using the coordinates $(u, v) \in \mathbb{R}^2$ with metric tensor $g = \cosh^2(v) du^2 + dv^2$ as above, we compute the Christoffel symbols of the Levi-Civita connection ∇. Since u, v are coordinates, the corresponding vector fields ∂_u, ∂_v commute: $[\partial_u, \partial_v] = 0$ and since ∇ has zero torsion,

$$\nabla_u \partial_v = \nabla_v \partial_u.$$

Since all the inner products of this basis are constant except for $g(\partial_u, \partial_u)$,

$$0 = \partial_u g(\partial_u, \partial v) = \partial_v g(\partial_u, \partial v) = \partial_u g(\partial_v, \partial v) = \partial_v g(\partial_v, \partial v)$$

and g is parallel with respect to ∇, implies

$$\nabla_v \partial_v = 0$$

and differentiating $g(\partial_u, \partial_u) = \cosh^2(v)$ yields

$$\nabla_u \partial_v = \nabla_v \partial_u = \tanh(v)\partial_u$$
$$\nabla_u \partial_u = \sinh(v)\cosh(v)\partial_v$$

One can modify this coordinate basis to an orhonormal basis by replacing ∂_u by

$$U := \operatorname{sech}(v)\partial_u,$$

in which the covariant derivatives are:

$$\begin{aligned} \nabla_U(U) &= -\tanh(v)\partial_v, \quad \nabla_v(U) = 0, \\ \nabla_U(\partial_v) &= \quad \tanh(v)U, \quad \nabla_v(\partial_v) = 0. \end{aligned}$$

6. Geodesic curvature in Fermi coordinates

Let $\gamma(t) = \big(u(t), v(t)\big)$ be a curve. Its velocity is:

$$\gamma'(t) = u'(t)\partial_u + v'(t)\partial_v$$

and the square of its speed is:

$$\left(\frac{ds}{dt}\right)^2 = u'(t)\cosh^2\big(v(t)\big) + v'(t)^2$$

Its acceleration is:

$$\begin{aligned} \frac{D}{dt}\gamma'(t) \;=\; & \Big(u''(t) + \tanh\big(v(t)\big)u'(t)v'(t)\Big)\partial_u \\ & + \Big(v''(t) + \tanh\big(v(t)\big)u'(t)v'(t) - u'(t)^2\cosh\big(v(t)\big)\sinh\big(v(t)\big)\Big)\partial_u \end{aligned}$$