A REMARKABLE FAMILY OF AFFINE CUBIC SURFACES

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Abstract. The family of affine cubics defined by

\[ x^2 + y^2 + z^2 - xyz = k + 2 \]

arises in several contexts, including relative \( \text{SL}(2, \mathbb{C}) \)-character varieties of the one-holed torus. We describe their geometry and symmetry.

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The trace \( \text{tr}[X,Y] \) of the commutator \([X,Y] = XYX^{-1}Y^{-1}\) of two elements \(X, Y\) of \(\text{SL}(2,\mathbb{C})\) defines a family of affine cubic surfaces

\[
S_k := \{(x,y,z) \in \mathbb{C}^3 \mid \kappa(x,y,z) = k\}
\]

where

\[
\kappa(x,y,z) := x^2 + y^2 + z^2 - xyz - 2
\]

and \(x = \text{tr}(X), y = \text{tr}(Y), z = \text{tr}(XY)\). In homogeneous coordinates \(X,Y,Z,W\) where

\[
(x, y, z) \mapsto \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}
\]

is the affine chart to the patch defined by \(W \neq 0\),

\[
x = X/W \\
y = Y/W \\
z = Z/W
\]

are affine coordinates. Its projective completion \(\overline{S_k} \subset \mathbb{P}^3\) is defined by

\[
(X^2 + Y^2 + Z^2)W - XYZ - (k + 2)W^3 = 0
\]

When \(k \neq \pm 2\), this surface is smooth. \(S_{-2}\) is the Markoff cubic, \(S_{-10/3}\) is the Fermat cubic, \(S_2\) is the Cayley cubic, and \(S_{18}\) is the Clebsch cubic. When \(k \geq 2\), all the lines are
real and we mainly concentrate on this case. (The limiting case $S_{\infty}$ is the triple plane $W^3 = 0$.)

We use the following notation and terminology: if $S \subset \mathbb{A}^3$ is an (affine) cubic surface, denote its closure in $\mathbb{P}^3$ by $\overline{S}$. A line on $S$ (respectively $\overline{S}$) is an affine (respectively projective) line contained in $S$ (respectively $\overline{S}$). The intersection of two lines on $\overline{S}$ will be called a crossing point. A crossing point is an Eckardt point (or an E-point) if it is the intersection of three distinct lines. A plane $P \subset \mathbb{A}^3$ is a tritangent plane (or simply a tritangent) if it is tangent to $S$ at three points or an Eckardt point. It is called generic if its intersection with $S$ is a union of three crossing lines; otherwise we call it an Eckardt tritangent. If $T$ is a generic tritangent, then it contains three crossing points (the intersection of three lines) and equals the tangent plane $T_pS$ for any of the crossing points $p$. If $T$ is an Eckardt tritangent, then $T \cap S$ is the union of three distinct lines intersecting at an Eckardt point $p$ and $T = T_pS$.

The generic smooth projective cubic surface $S$ over $\mathbb{C}$ contains 27 lines, 45 tritangents and 135 crossing points (and no E-points). The cubic surfaces in our family contains a generic tritangent $T$ which contains 6 E-points.

1. Three ideal lines in the ideal plane

We begin by discussing the tritangent plane at infinity, which is generic, in that it three lines in general position. Thus the 27 lines divide into 3 ideal lines and 24 finite lines. The first step in our classification of lines uses these three lines to divide the finite lines into three 8-line families.

1.1. Locus at infinity. The structure we consider is that of the affine cubic $S_k$, or, more precisely, the pair $(\overline{S}_k, \mathcal{T}_\infty)$ where $\mathcal{T}_\infty$ is the plane $W = 0$ of ideal points in the fixed affine patch of $\mathbb{P}^3$. The ideal plane $\mathcal{T}_\infty$ is a generic tritangent plane, that is, the tangent plane to $\overline{S}_k$ at three distinct points

\[
x_{p_\infty} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad y_{p_\infty} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad z_{p_\infty} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\]
\( \mathcal{T}_\infty \) intersects \( \overline{S}_k \) in three crossing lines \( XYZ = W = 0 \),

\[
\begin{align*}
X \mathcal{I} & := \{ X = W = 0 \} \\
Y \mathcal{I} & := \{ Y = W = 0 \} \\
Z \mathcal{I} & := \{ Z = W = 0 \}.
\end{align*}
\]

The *ideal locus* \( \overline{S}_k \setminus S_k \) is their union. These lines intersect in the points \( X_\infty, Y_\infty, Z_\infty \), which are the singular points of \( \overline{S}_k \setminus S_k \). The ideal plane \( \mathcal{T}_\infty \) is tangent to \( \overline{S}_k \) at these three points.

Although \( \overline{S}_k \) is smooth at \( X_\infty, Y_\infty, Z_\infty \), it does *not* intersect the ideal plane \( \{ W = 0 \} \) transversely, and the ideal locus

\[
\overline{S}_k \setminus S_k = \overline{S}_k \cap \mathcal{T}_\infty = X \mathcal{I} \cup Y \mathcal{I} \cup Z \mathcal{I}
\]

is not a manifold. In particular the points \( X_\infty, Y_\infty, Z_\infty \) are ideal crossing points of \( \overline{S}_k \).

Our context is thus the family of projective cubics \( \overline{S}_k \) together with a fixed generic tritangent plane; see Goldman-Toledo [3]. These pairs \( (\overline{S}_k, \mathcal{T}_\infty) \) also exhibit extra symmetry, which fails in general. This symmetry is reflected in the absence of linear terms in \( \kappa \), which is the defining function for \( S_k \). Alternatively, it is reflected in the existence of ideal Eckardt points (8). These linear terms do occur in the analogous description of the relative character variety of the 4-holed sphere, where the \( \mathbb{Z}/2 \oplus \mathbb{Z}/2 \)-symmetry is broken.

1.2. Three families of non-ideal lines. The basic fact we exploit is the following:

**Theorem 1.2.1.** *On each line \( \ell \subset S_k \), exactly one of the coordinate functions \( x, y, z \) is constant.*

Thus the finite lines fall into three families labeled by \( X, Y, \) or \( Z \) respectively. Each of the three families consists of of eight lines, grouped into four pairs.

We use the elementary fact:

**Lemma 1.2.2.** *Suppose that \( \mathbb{A}^3 \xrightarrow{f} \mathbb{k} \) is an affine function and \( \ell \subset \mathbb{A}^3 \) is a line. Let \( F \) be the covector in \( (\mathbb{k}^3)^* \) extending \( f \) and let \( \lambda \) be the line in the ideal plane \( P_\infty \) corresponding to \( \ker(F) \). Then the ideal point of \( \ell \) lies in \( \lambda \) if and only if the function \( f \) is constant on \( \ell \).*

**Proof.** Theorem 1.2.1 follows easily from Lemma 1.2.2 as follows. Each line \( \ell \) on \( S_k \) has an ideal point

\[
\ell_\infty := \ell \cap \mathcal{T}_\infty.
\]
Since the ideal locus
\[ S_k \cap T_{\infty} = XI \cup YI \cup ZI, \]
the ideal point \( \ell_{\infty} \) must lie in an ideal coordinate line. Then \( \ell_{\infty} \in XI \) if and only if the affine coordinate \( x \) is constant on \( \ell \) (and similarly for \( y \) and \( z \)).

A more pedestrian and intuitive way of seeing Theorem 1.2.1 is to imagine a line on \( S_k \) given in parametric form by:
\[ t \mapsto p(t) = (x_0 + t\xi, y_0 + t\eta, z_0 + t\zeta). \]
The composition \( \kappa \circ p(t) \) is constant, yet it is given by a cubic polynomial in \( t \) with leading term \(-t^3\xi\eta\zeta\). Letting \( t \mapsto \infty \), we see that \( \xi\eta\zeta = 0 \), that is, one of the three coordinates \( x, y, z \) is constant on \( p(t) \).

For example, the family corresponding to the \( Z \)-coordinate yields four planes in \( \mathbb{A}^3 \) defined by
\[ z_0 = -\sqrt{k+2}, \]
\[ z_0 = -2, \]
\[ z_0 = +2 \]
\[ z_0 = +\sqrt{k+2}. \]
(1)
Each plane contains two lines, and the union with the ideal line \( Z_{\infty} \) is a tritangent. The lines in the planes \( z_0 = \pm\sqrt{k+2} \) we label with “\( C \)” (for “crossing”). The lines in the planes \( z_0 = \pm 2 \) we label with “\( P \)” for (for “parallel”). In general the intersection of \( S_k \) with plane \( z = z_0 \) is a conic, but this conic degenerates into a union of lines at the special levels (1). Thus each special level contains two lines, either parallel (\( P \)) or crossing (\( C \)).

This follows easily from writing the defining equation in terms of the family of quadratic forms:
\[ Q_z(x, y) := x^2 - zxy + y^2 \]
Then \( S_k \) is defined by:
\[ Q_z(x, y) = k + 2 - z^2 \]
(3)
because \( \kappa(x, y, z) = Q_z(x, y) - z^2 = 2 \). When \( z = \pm 2 \),
\[ Q_2(x, y) = (x - y)^2 \]
\[ Q_{-2}(x, y) = (x + y)^2. \]
(4)
When \( z = \pm \sqrt{k + 2} \),

\[
Q_{\sqrt{k+2}}(x,y) = (y - m^+ x)(y - m^- x) \\
Q_{-\sqrt{k+2}}(x,y) = (y + m^+ x)(y + m^- x)
\]

(5)

where the two slopes, \( m^* \in \mathbb{Q}[\sqrt{k + 2}, \sqrt{k - 2}] \) are defined by:

\[
m^* := \frac{\sqrt{k + 2} \pm \sqrt{k - 2}}{2}
\]

(6)

The slopes satisfy:

\[
m^* m^- = 1, \quad m^+ + m^- = \sqrt{k + 2}, \quad m^+ - m^- = \sqrt{k - 2}.
\]

2. **Singular points and symmetry**

Next we show that \( S_k \) is smooth when \( z \neq \pm 2 \). Then we discuss the group of automorphisms of \( \kappa \). Automorphisms extending to the projective completion \( \overline{S_k} \) form the finite group of linear automorphisms, isomorphic to the symmetric group \( S_4 \).

The family \((\overline{S_k}, S_k)\) enjoys the symmetry of \( S_3 \), since the polynomial \( \kappa(x, y, z) \) is symmetric. For listing the geometric objects, we exploit the 3-cycles in the cyclic alternating group

\[
\mathfrak{A}_3 = \{ ( ), (123), (132) \} \leq S_3.
\]

2.1. **Critical points of \( \kappa \)**. The critical values of \( \kappa \) are \( \pm 2 \). The critical points of \( \kappa \) are:

- The **origin** (for critical value \(-2\))
  \[
o := (0,0,0),
\]

- and the **vertices** (for critical value \(+2\))
  \[
c_0 := (2,2,2); \\
c_1 := (2,-2,-2); \\
c_2 := (-2,2,-2); \\
c_3 := (-2,-2,2).
\]

(7)

They constitute the singular sets of the Markoff cubic \( S_{-2} \) and the Cayley cubic \( S_2 \) respectively. They are all **nodes** (ordinary double points) of the respective level sets.
The origin \( o = (0, 0, 0) \) is the character of the quaternion representation, given by Pauli matrices
\[
\begin{bmatrix}
i & 0 \\0 & -i
\end{bmatrix}, \begin{bmatrix}0 & -1 \\1 & 0
\end{bmatrix}, \begin{bmatrix}0 & i \\i & 0
\end{bmatrix}.
\]

The vertices
\[
\text{Sing}(S_{+2}) = \mathcal{C} := \{c_0, c_1, c_2, c_3\}
\]
are characters of central representations, that is, representations \( F_2 \rightarrow \{\pm I_2\} \). These representations form a group
\[
\Delta := \text{Hom}(F_2, \{\pm I_2\}) \cong (\mathbb{Z}/2 \oplus \mathbb{Z}/2),
\]
acting simply transitively on \( \mathcal{C} \). In particular
\[
\begin{align*}
c_0 & \xrightarrow{\sigma_1} c_1, & c_2 & \xrightarrow{\sigma_1} c_3 \\
c_0 & \xrightarrow{\sigma_2} c_2, & c_3 & \xrightarrow{\sigma_2} c_1 \\
c_0 & \xrightarrow{\sigma_3} c_3, & c_1 & \xrightarrow{\sigma_3} c_2.
\end{align*}
\]
and thus correspond to free involutions in \( \mathfrak{S}_4 = \text{Aut}(\mathcal{C}) \) via:
\[
\begin{align*}
\sigma_1 & : (01)(23) \\
\sigma_2 & : (02)(13) \\
\sigma_3 & : (03)(12)
\end{align*}
\]
As automorphisms of the family of cubics in \( \mathbb{C}^3 \), these are represented by sign-change automorphisms in \( \text{SL}(3, \mathbb{Z}) \):
\[
\begin{align*}
\sigma_1 & := \begin{bmatrix}1 & 0 & 0 \\0 & -1 & 0 \\
0 & 0 & -1
\end{bmatrix}, & \sigma_2 & := \begin{bmatrix}-1 & 0 & 0 \\0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}, & \sigma_3 & := \begin{bmatrix}-1 & 0 & 0 \\0 & -1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\end{align*}
\]
Their fixed points on the cubic level sets \( S_k \) are the dihedral characters or \textit{D-points}
\[
\begin{align*}
^X\text{Dih}^\pm & = (\pm \sqrt{k+2}, 0, 0) \\
^Y\text{Dih}^\pm & = (0, \pm \sqrt{k+2}, 0) \\
^Z\text{Dih}^\pm & = (0, 0, \pm \sqrt{k+2})
\end{align*}
\]
corresponding to dihedral representations. For example, \( D_{k}^+ = (0, 0, \sqrt{k + 2}) \) corresponds to:

\[
\begin{bmatrix}
0 & -1 \\
1 & 0
\end{bmatrix},
\begin{bmatrix}
0 & \sqrt{k + 2} \\
-1/\sqrt{k + 2} & 0
\end{bmatrix},
\begin{bmatrix}
m^- & 0 \\
0 & m^+
\end{bmatrix}
\]

where the slopes \( m^\pm \) are defined in (6).

These corresponds to 6 crossing points when \( k \neq -2 \), which coalesce to the origin \( o \) as \( k \to -2 \).

3. Symmetry

We first describe the automorphisms of these cubics. The \textit{linear automorphisms} form the symmetric group \( S_4 \), which extend to projective automorphisms of \( S_k \). We first describe this action, which combines the triple symmetry of the three variables with the sign-change group described above. The group \( S_4 \) arises concretely as automorphisms of several natural 4-element sets:

- The \textit{vertices}, that is, the \textit{nodes} of the Cayley cubic \( S_2 \);
- The ends of the real locus \( \mathbb{R}^3 \cap S_k \);
- Complementary ideal triangular regions \( S_k \), that is, the four components of the complement of the ideal locus

\[
X \mathcal{I} \cup Y \mathcal{I} \cup Z \mathcal{I} \subset \mathcal{T}_\infty.
\]

in the ideal tritangent plane.
- The finite \( E \)-points \( \{ \text{Eck}_0, \text{Eck}_1, \text{Eck}_2, \text{Eck}_3 \} \) of \( S_{18} \).

3.1. Linear automorphisms. These cubics all exhibit a finite symmetry group \( S_4 \), which can be realized as the group of \textit{all} automorphisms of the four-element set \( \mathcal{C} \), which are the vertices of a natural \textit{tetrahedron} on the Cayley cubic \( S_2 \).

Observe first that these cubics are symmetric in the three coordinates \( X, Y, Z \), leading to an action of the symmetric group \( S_3 \), which we will heavily exploit. The permutations of coordinates lead to permutations of \( \mathcal{C} \) which fix \( c_0 \) but permute \( \{ c_1, c_2, c_3 \} \). The rest of \( S_4 \) can be understood in terms of the group \( \Delta \) of \textit{sign-change automorphisms}, described above.

In terms of coordinates, the symmetric group is a split extension

\[
\Delta < S_4 \to S_3,
\]

where the epimorphism \( S_4 \to S_3 \) is realized by permuting the three coordinates, and the kernel \( \Delta \) comprises sign-changes as above. For representations, the sign-changes correspond to the action of the group of
central representations $\text{Hom}(F_2, \{ \pm 1 \})$ on $\text{Hom}(F_2, \text{SL}(2, \mathbb{C}))$. The orbits comprise lifts of $\text{PSL}(2, \mathbb{C})$-representations to $\text{SL}(2, \mathbb{C})$ and the relative character variety $S_k$ corresponds to the image of a $\Delta$-$\text{Inn}(\text{SL}(2, \mathbb{C}))$-invariant subset of $\text{Hom}(F_2, \text{SL}(2, \mathbb{C}))$ under the $\text{Inn}(\text{SL}(2, \mathbb{C}))$-quotient map.

The symmetric group $S_4$ is the group of automorphisms of the projective cubic $S_k$ for generic $k$, and is realized as above by linear automorphisms of $S_k$. Both the ends of the real level sets $\overline{S_k} \cap \mathbb{R}^3$...

Another finite subset invariant under the automorphism group $S_4$ is the six-element subset of ideal Eckardt points defined in (8). These are the ideal points of the $P$-lines.

As for the six-element subset comprising dihedral characters, $S_4$ is the centralizer of an involution in $S_6$ corresponding to the six-element subset consisting of ordered pairs of distinct points of $\{0, 1, 2, 3\}$.

### 3.2. Galois automorphisms.

The lines, tritangent planes, and their intersections also enjoy Galois symmetry as follows. Their coordinates lie in the biquadratic field $\mathbb{Q}[\sqrt{k+2}, \sqrt{k-2}]$, at least when $\sqrt{k \pm 2} \notin \mathbb{Q}$. Its Galois group is generated by involutions

$$\sqrt{k+2} \leftrightarrow -\sqrt{k+2}, \quad \sqrt{k-2} \leftrightarrow -\sqrt{k-2}.$$  

This group, also isomorphic to $(\mathbb{Z}/2 \oplus \mathbb{Z}/2)$, acts on the configuration of lines, tritangents, and crossing points.

Each pair of lines in one of the three coordinate families ($X, Y$ or $Z$) and four levels (1) is interchanged by the Galois involution $\mathcal{S}^{-}$. Note that (6) implies that

$$m^\pm \leftrightarrow m^\mp, \quad m^\pm \leftrightarrow -m^\mp$$

and $\mathcal{S}^+ \circ \mathcal{S}^- = \mathcal{S}^- \circ \mathcal{S}^+$ is an involution interchanging $m^\pm$ and $-m^\pm$.

### 3.3. Vieta involutions.

In addition to the finite groups of automorphisms which extend to projective automorphisms, the affine cubics $S_k$ admit infinite groups of symmetries which define interesting dynamical systems. Namely, the coordinate projections $\mathbb{C}^3 \longrightarrow \mathbb{C}^2$ define double (branched coverings) of $S_k$, and their Galois groups generate an action of the free 3-generator Coxeter group $\mathbb{Z}/2 \ast \mathbb{Z}/2 \ast \mathbb{Z}/2$.

Take, for example, the coordinate projection for the $z$-coordinate. Fix $x_0, y_0 \in \mathbb{C}$. Then the restriction of the defining cubic polynomial $\kappa$ to the coordinate line $\{(x_0, y_0)\} \times \mathbb{C}$ is quadratic. Thus the intersection

$$S_k \cap \{(x_0, y_0)\} \times \mathbb{C}$$
corresponds to the pair of solutions \( z \) of the quadratic equation

\[
k = \kappa(x_0, y_0, z) = z^2 - (x_0y_0)z + (x^2 + y^2 - 2).
\]

If \( z, z' \) are the two solutions, then \( z + z' = x_0y_0 \), so

\[z' = x_0y_0 - z.\]

The deck transformation of the double covering \( S_k \to \mathbb{C}^2 \) is the Vieta \textit{involution}. The three Vieta involutions are:

- \( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \xrightarrow{x_\nu} \begin{bmatrix} x' := yz - x \\ y \\ z \end{bmatrix} \)
- \( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \xrightarrow{y_\nu} \begin{bmatrix} x \\ y' := zx - y \\ z \end{bmatrix} \)
- \( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \xrightarrow{z_\nu} \begin{bmatrix} x \\ y \\ z' := xy - z \end{bmatrix} \)

They generate a free Coxeter group \( \mathbb{Z}/2 \ast \mathbb{Z}/2 \ast \mathbb{Z}/2 \), naturally isomorphic to the level 2 congruence subgroup \( \text{PGL}(2, \mathbb{Z})_{(2)} \). Specifically, the respective Vieta involutions \( z_\nu, x_\nu, y_\nu \) are realized by the automorphisms of \( \mathbb{P}_2 \) and the corresponding elements of \( \text{PGL}(2, \mathbb{Z}) \) acting on points \( z \) in the upper half-plane, respectively:

- \( \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \xrightarrow{z_\nu} \begin{bmatrix} X \\ Y^{-1} \\ YX^{-1} \end{bmatrix} \leftrightarrow \pm \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} : (z \mapsto -\overline{z}) \)
- \( \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \xrightarrow{x_\nu} \begin{bmatrix} Y^{-1}X^{-1}Y^{-1} \\ Y \\ Z^{-1} \end{bmatrix} \leftrightarrow \pm \begin{bmatrix} -1 & 0 \\ -2 & 1 \end{bmatrix} : (z \mapsto \frac{\overline{z}}{1 - 2\overline{z}}) \)
- \( \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \xrightarrow{y_\nu} \begin{bmatrix} X^{-1} \\ X^2Y^{-1} \\ Z \end{bmatrix} \leftrightarrow \pm \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} : (z \mapsto 2 - \overline{z}). \)

The realizations in \( \text{PGL}(2, \mathbb{Z}) \) are reflections in the imaginary axis \( i\mathbb{R}_+ \), the unit semicircle \(|z| = 1\), and the vertical line \( 1 + i\mathbb{R}_+ \), respectively. (Compare [4].)
4. Two types of non-ideal lines

We classify the 24 non-ideal lines into 12 \( P \)-lines and 12 \( C \)-lines. Furthermore, as in §1.2, these fall into three families, corresponding to the coordinates \( X, Y, Z \). Each family contains 4 \( P \)-lines and 4 \( C \)-lines.

4.1. \( P \)-lines. The \( P \)-lines arise when these lines are parallel, namely when \( x_0 = \pm 2 \), \( y_0 = \pm 2 \), or \( z_0 = \pm 2 \) respectively. The four \( P \)-lines in the \( Z \)-family are:

\[
\begin{align*}
&zP^+_z: \quad z = +2 \quad y = x + \sqrt{k-2} \\
&zP^-_z: \quad z = +2 \quad y = x - \sqrt{k-2} \\
&zP^+_z: \quad z = -2 \quad y = -x + \sqrt{k-2} \\
&zP^-_z: \quad z = -2 \quad y = -x - \sqrt{k-2}
\end{align*}
\]

arising from the factorizations (4). Apply 3-cycles in \( A_3 \) to obtain similar formulas for the \( X \)-family and the \( Y \)-family of \( P \)-lines.

These lines fall into six pairs of parallelism classes, namely \( xP^+_x \), \( yP^+_y \), and \( zP^+_z \) respectively. They meet in six ideal Eckardt points:

\[
\begin{align*}
&E^+_x \coloneqq xP^+_x \cap xP^-_x = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} & &E^-_x \coloneqq xP^-_x \cap xP^-_x = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \\
&E^+_y \coloneqq yP^+_y \cap yP^-_y = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} & &E^-_y \coloneqq yP^-_y \cap yP^-_y = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \\
&E^+_z \coloneqq zP^+_z \cap zP^-_z = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} & &E^-_z \coloneqq zP^-_z \cap zP^-_z = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}
\end{align*}
\]
4.2. \( C \)-lines. \( C \)-lines arise when the plane section is a degenerate conic, consisting of a pair of crossing lines. Consider the \( C \)-lines in the \( Z \)-family. Rewrite the defining equation for \( S_k \) as (3) using the family of quadratic forms \( Q_z \), defined in (2). Thus the \( C \)-lines occur at the levels 

\[ z = z_0 \text{ where } Q_{z_0}(x, y) = 0, \text{ that is, when } z^2 = k + 2. \]

On these levels, the factorization (5) implies that the plane section is the union of lines 

\[ y = m^\pm x \]

where the slopes \( m^\pm \) are defined in (6). Similarly on the level \( z_0 = -\sqrt{k + 2} \), the conic defined by \( Q_{z_0} \) degenerates into the pair of lines 

\[ y = -m^\pm x \]

Thus the four \( C \)-lines in the \( Z \)-family are:

\[
\begin{align*}
ZC_+^+ & : \quad z = +\sqrt{k + 2}, \quad y = m^+ x \\
ZC_+^- & : \quad z = +\sqrt{k + 2}, \quad y = m^- x \\
ZC_-^+ & : \quad z = -\sqrt{k + 2}, \quad y = -m^+ x \\
ZC_-^- & : \quad z = -\sqrt{k + 2}, \quad y = -m^- x
\end{align*}
\]

As with \( P \)-lines, these lines naturally pair; however the \( C \)-lines cross rather than being parallel: For example, the two lines \( ZC_+^+ \) and \( ZC_-^+ \) cross, meeting in the crossing point 

\[ Z\text{Dih}^+ = (0, 0, \sqrt{k + 2}), \]

which we call a dihedral character, or a D-point, since it corresponds to the “dihedral” representation with \( \rho(X), \rho(Y) \) symmetries in points and \( \rho(Z) \) a transvection along the geodesic joining \( \text{Fix}(\rho(X)) \) and \( \text{Fix}(\rho(Y)) \).

These lines, together with \( ZI \), span the tritangent plane extending \( z = +\sqrt{k + 2} \). Together with \( ZI \), they span the tritangent plane extending \( z = -\sqrt{k + 2} \).

4.3. Linear automorphisms and lines. We now describe the action of \( \mathfrak{S}_4 \) on the set of lines.

A sign-change leaves invariant each line in the corresponding family of \( C \)-lines. Then, for example the lines in the \( Z \)-family

\[
\begin{align*}
ZC_+^+ & = \{z = +\sqrt{k + 2}, \ y = m^+ x\} \\
ZC_-^+ & = \{z = -\sqrt{k + 2}, \ y = -m^+ x\}
\end{align*}
\]
are each invariant under \((x, y, z) \mapsto (-x, -y, z)\) which corresponds to the transposition \(\sigma^{(03)} \in \mathfrak{S}_4\). The transposition \(\sigma^{(12)}\) interchanges
\[ Z\mathcal{C}_+ \leftrightarrow Z\mathcal{C}_-. \]
The two remaining sign-changes (corresponding to \(\sigma^{(01)(23)}\) and \(\sigma^{(02)(13)}\) in \(\mathfrak{S}_4\)) interchange
\[ Z\mathcal{C}_+^- \leftrightarrow Z\mathcal{C}_-^+.\]
Here is what happens for \(\mathcal{P}\)-lines. Consider the \(Z\)-family. Each
\[ Z\mathcal{P}_-^+ = \{z = -2, x + y = \pm \sqrt{k - 2}\} \]
is invariant under the transposition \((12)\). The sign-change \(\sigma_3\) interchanges these two lines; the other sign-changes interchange
\[ Z\mathcal{P}_+^- \leftrightarrow Z\mathcal{P}_+^+. \]
For \(Z\mathcal{P}_+^- = \{z = 2, y - x = \pm \sqrt{k - 2}\}\), the transposition
\[ (03) = \sigma_3(12) = (12)\sigma_3 = \sigma_1(12)\sigma_1 = \sigma_2(12)\sigma_2 \in \mathfrak{S}_4 \]
is realized by \((x, y, z) \mapsto (-y, -x, z)\) and leaves each of these two lines invariant.

4.4. Linear involutions and lines. Indeed, the involutions in \(\mathfrak{S}_4\) distinguish the \(\mathcal{P}\)-lines from the \(\mathcal{C}\)-lines as follows. The alternating group \(\mathfrak{A}_4 < \mathfrak{S}_4\) consists of even permutations. The nontrivial elements of the sign-change group \(\Sigma < \mathfrak{A}_4\) consists of even permutations of order two, namely products of disjoint transpositions
\[ \sigma^{(ij)(14)} \leftrightarrow \sigma_l \]
where \(\{i, j, l\} = \{1, 2, 3\}\). The odd permutations of order two are the transpositions \(\sigma^{(ij)}\) where \(\{i, j\}\) is a 2-element subset of \(\{1, 2, 3\}\) and
\[ \sigma^{(14)} \leftrightarrow \sigma_1 \circ \sigma^{(ij)} \].

**Proposition 4.4.1.** Let \(\ell \in \mathcal{S}_k\) be a (non-ideal) line. Then \(\ell\) is invariant under an odd involution \(\Longleftrightarrow \ell\) is a \(\mathcal{P}\)-line, and \(\ell\) is invariant under an even involution \(\Longleftrightarrow \ell\) is a \(\mathcal{C}\)-line.

The action of \(\mathfrak{S}_4\) on the ideal lines in \(\overline{\mathcal{S}_k}\) is even easier. Each sign-change in \(\Delta\) fixes each ideal line pointwise. The subgroup
\[ \mathfrak{S}_3 = \text{Aut}(\{1, 2, 3\}) < \mathfrak{S}_4 \]
complementary to \(\Delta \triangleleft \mathfrak{S}_4\) acts by permutations of \(\{X\mathcal{I}, Y\mathcal{I}, Z\mathcal{I}\}\).
4.5. Vieta automorphisms and lines. The Vieta involutions do not extend to projective space (they are not even defined by homogeneous polynomials), and therefore do not act on the ideal lines. We describe their action on a sample $\mathcal{P}$-line and a sample $\mathcal{C}$-line. As $\mathfrak{S}_4$ acts transitively on the set of $\mathcal{P}$-lines (respectively $\mathcal{C}$-lines), it suffices for the discussion to consider one sample line from each type.

First consider the $\mathcal{P}$-line $\mathbb{Z}^+$. The action of $\nu_X$ on it is:

\[
\begin{bmatrix}
x \\
x - \sqrt{k - 2} \\
2
\end{bmatrix} \mapsto \begin{bmatrix}
x - 2\sqrt{k - 2} \\
(x - 2\sqrt{k - 2}) + 2\sqrt{k - 2} \\
2
\end{bmatrix}
\]

and the $\nu_Y$-action is:

\[
\begin{bmatrix}
x \\
x - \sqrt{k - 2} \\
2
\end{bmatrix} \mapsto \begin{bmatrix}
x \\
x + \sqrt{k - 2} \\
2
\end{bmatrix}
\]

Since both involutions map $\mathbb{Z}^+ \mathcal{P} \to \mathbb{Z}^+ \mathcal{P}$, their composition $\nu_X \circ \nu_Y$ preserves each $\mathbb{Z}^+ \mathcal{P}$, translating by $-\sqrt{k - 2}$ on each.

The involution $\nu_Z$ maps $\mathbb{Z}^+ \mathcal{P}$ to a parabola:

\[
\begin{bmatrix}
x \\
x - \sqrt{k - 2} \\
2
\end{bmatrix} \mapsto \begin{bmatrix}
x \\
(x + \sqrt{k - 2})^2 - \frac{k+6}{4}
\end{bmatrix}
\]

Next consider the $\mathcal{C}$-line $\mathbb{Z}^+ \mathcal{C}$. Both $\nu_X$ and $\nu_Y$ map $\mathbb{Z}^+ \mathcal{C} \to \mathbb{Z}^+ \mathcal{C}$:

\[
\begin{bmatrix}
x \\
m^+x + \sqrt{k + 2}
\end{bmatrix} \xrightarrow{\nu_X} \begin{bmatrix}
(m^+)^2x \\
m^+x + 2\sqrt{k + 2}
\end{bmatrix}
\]

\[
\begin{bmatrix}
x \\
m^+x + \sqrt{k + 2}
\end{bmatrix} \xrightarrow{\nu_Y} \begin{bmatrix}
x \\
m^-x + \sqrt{k + 2}
\end{bmatrix}
\]

Since both involutions map $\mathbb{Z}^+ \mathcal{C} \to \mathbb{Z}^+ \mathcal{C}$, their composition $\nu_X \circ \nu_Y$ preserves $\mathbb{Z}^+ \mathcal{C}$, scaling by $(m^-)^2$.
The involution $z_\nu$ maps $\mathcal{C}^+_\mp$ to a parabola:

$$\begin{bmatrix}
x \\
m^+_x \\
+\sqrt{k+2}
\end{bmatrix} \longmapsto \begin{bmatrix}
x \\
m^+_x \\
-m^+_x^2 - \sqrt{k+2}
\end{bmatrix}$$

4.6. **Degeneration of lines: the Markoff surface** $k = -2$. When $k \to -2$, the levels at $\pm \sqrt{k+2}$ coalesce at the level 0, and each pair $\mathcal{C}^+_\pm, \mathcal{C}^-\pm$ converge to a single line, for example:

$$\mathcal{X}_\pm := \{ z = \pm iy, x = 0. \}$$
$$\mathcal{Y}_\pm := \{ x = \pm iz, y = 0. \}$$
$$\mathcal{Z}_\pm := \{ y = \pm ix, z = 0. \}$$

This gives 6 $\mathcal{C}$-lines, each counted with multiplicity 2. The remaining 12 $\mathcal{P}$-lines are:

$$\mathcal{X}_\pm \mp : = \{ z = y \pm 2i, x = \mp 2 \}$$
$$\mathcal{Y}_\pm : = \{ x = z \pm 2i, y = \mp 2 \}$$
$$\mathcal{Z}_\pm : = \{ y = x \pm 2i, z = \mp 2 \}$$

This gives 6 double $\mathcal{C}$-lines, 12 $\mathcal{P}$-lines and 3 ideal lines, verifying the total count of 27 lines with multiplicity. The singularity at $o$ is the concurrent intersection of three double lines.

4.7. **Degeneration of lines: the Cayley surface** $k = 2$. The degeneration is more severe on $S_{+2}$. In that case, $\sqrt{k-2} = 0$ implies that all the $\mathcal{P}$-lines $\mathcal{P}^\pm_\pm$ (respectively $\mathcal{P}^\pm_{\pm}$) coalesce. Furthermore since $m^+ = m^- = 1$, the $\mathcal{C}$-lines $\mathcal{C}^\pm_\pm$ (respectively $\mathcal{C}^\pm_{\pm}$) coalesce. There remain 6 quadruple lines:

$$\mathcal{X}_\mp : = \{ z = y, x = \mp 2 \}$$
$$\mathcal{X}_\mp : = \{ z = -y, x = \mp 2 \}$$
$$\mathcal{Y}_\mp : = \{ x = z, y = \mp 2 \}$$
$$\mathcal{Y}_\mp : = \{ x = -z, y = \mp 2 \}$$
$$\mathcal{Z}_\mp : = \{ y = x, z = \mp 2 \}$$
$$\mathcal{Z}_\mp : = \{ y = -x, z = \mp 2 \}$$
5. Galois Automorphisms

The lines, tritangent planes, and their intersections also enjoy Galois symmetry as follows. Their coordinates lie in the biquadratic field \( \mathbb{Q}[\sqrt{k+2}, \sqrt{k-2}] \), at least when \( \sqrt{k \pm 2} \notin \mathbb{Q} \). Its Galois group is generated by involutions

\[
\sqrt{k+2} \xleftrightarrow{G^+} -\sqrt{k+2}, \quad \sqrt{k-2} \xleftrightarrow{G^-} -\sqrt{k-2}.
\]

This group, also isomorphic to \( \left( \mathbb{Z}/2 \oplus \mathbb{Z}/2 \right) \), acts on the configuration of lines, tritangent planes, and intersection points.

Each pair of lines in one of the three coordinate families \((X,Y)\) or \(Z\) and four levels \((1)\) is interchanged by the Galois involution \(G^-\) (see below). Observe that

\[
m^\pm \xleftrightarrow{G^-} m^\mp, \quad m^\pm \xleftrightarrow{G^+} -m^\mp
\]

and \(G^+ \circ G^- = G^- \circ G^+\) is an involution interchanging \(m^\pm\) and \(-m^\mp\).

5.1. Galois Automorphisms and Lines. We observe that the Galois automorphisms act on the \(P\)-lines and \(C\)-lines as follows. The involution \(G^-\) interchanges the two lines in each pair of parallel \(P\)-lines, that is,

\[
P^+ \rightarrow P^-.
\]

However it interchanges the two slopes \(m^\pm\) so it also interchanges the two line each pair of crossing \(C\)-lines.

The involution \(G^+\), on the other hand, interchanges \(\sqrt{k+2} \leftrightarrow -\sqrt{k+2}\) and \(m^\pm \leftrightarrow -m^{mp}\) so it takes

\[
C_+ \leftrightarrow C_-.
\]

6. Example of a Double-Six

Recall that a Schlafli double-six consists of two ordered sextuples of lines \((a_1, \ldots, a_6)\) and \((b_1, \ldots, b_6)\) such that:

- If \(i \neq j\), then \(a_i \not\parallel a_j\).
- If \(i \neq j\), then \(b_i \not\parallel b_j\).
- \(a_i \not\parallel b_j\) if and only if \(i = j\).
- \(a_i\) and \(b_j\) intersect whenever \(i \neq j\).

In the last case \(a_i\) and \(b_j\) lie in a tritangent plane which we denote \(\mathcal{T}_{ij}\). Write \(p_{ij}\) for the point of intersection \(a_i \cap b_j\); necessarily \(p_{ij} \in \mathcal{T}_{ij}\). Furthermore \(\mathcal{T}_{ij}\) meets \(S_k\) in a third line, which we denote \(c_{ij}\).

I am grateful to Damiano Testa for supplying the proof of the following fact:

**Lemma 6.0.1.** If \(i \neq j\), then \(c_{ij} = c_{ji}\).
Proof. Since \( a_i \not\parallel a_j \) and \( a_j \not\parallel b_j \), it follows that \( p_{ij} \notin a_i \cup b_j \). Since \( \mathcal{T}_{ij} \) is a tritangent plane to \( S_k \), and intersects \( S_k \) in 
\[ a_i \cup b_j \cup c_{ij}, \]
\( p_{ij} \in c_{ij} \). In particular \( a_j \) intersects \( c_{ij} \).
Similarly, \( b_i \) intersects \( c_{ij} \). Since \( a_j, b_i, c_{ij} \in \text{Lines}(S_k) \), and they mutually cross, \( c_{ij} \in \mathcal{T}_{ji} \). Since these lines are distinct, \( c_{ij} = c_{ji} \). □

Although \( c_{ij} = c_{ji} \), the tritangent planes are distinct: \( \mathcal{T}_{ij} \neq \mathcal{T}_{ji} \).

Here is a simple example of a double-six:

\[
\begin{align*}
  a_1 &= X^+C^+ + Y^-C^+ + Z^-C^- + U_0, \\
  a_2 &= Y^+C^+ + X^-C^- + Z^-C^- + U_1, \\
  a_3 &= Y^+C^+ + X^-C^- + Z^-C^- + U_2, \\
  a_4 &= Y^+C^+ + X^-C^- + Z^-C^- + U_3, \\
  a_5 &= Y^+C^+ + X^-C^- + Z^-C^- + U_4.
\end{align*}
\]

The \( c_{ij} \) may be computed from the following table.

Clebsch’s *Diagonal Cubic Surface* is the intersection of the cubic hypersurface in \( \mathbb{CP}^3 \)
\[
(U_0)^3 + (U_1)^3 + (U_2)^3 + (U_3)^3 + (U_4)^3 = 0,
\]
with the hyperplane
\[
U_0 + U_1 + U_2 + U_3 + U_4 = 0.
\]
It clearly enjoys \( \mathfrak{S}_5 \)-symmetry, which is one of the maximal symmetry groups of a smooth projective cubic surface (\( \#\mathfrak{S}_5 = 5! = 120 \)).
Table 2. The Principal Double-Six

<table>
<thead>
<tr>
<th></th>
<th>$X^C$</th>
<th>$X^C$</th>
<th>$Y^C$</th>
<th>$Y^C$</th>
<th>$Z^C$</th>
<th>$Z^C$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X^C$</td>
<td>$X^I$</td>
<td>$Z^P$</td>
<td>$Z^P$</td>
<td>$Y^P$</td>
<td>$Y^P$</td>
<td></td>
</tr>
<tr>
<td>$X^C$</td>
<td>$X^I$</td>
<td>$Z^P$</td>
<td>$Z^P$</td>
<td>$Y^P$</td>
<td>$Y^P$</td>
<td></td>
</tr>
<tr>
<td>$Y^C$</td>
<td>$Z^P$</td>
<td>$Z^P$</td>
<td>$Y^I$</td>
<td>$X^P$</td>
<td>$X^P$</td>
<td></td>
</tr>
<tr>
<td>$Y^C$</td>
<td>$Z^P$</td>
<td>$Z^P$</td>
<td>$Y^I$</td>
<td>$X^P$</td>
<td>$X^P$</td>
<td></td>
</tr>
<tr>
<td>$Z^C$</td>
<td>$Y^P$</td>
<td>$Y^P$</td>
<td>$X^P$</td>
<td>$X^P$</td>
<td>$Z^I$</td>
<td></td>
</tr>
<tr>
<td>$Z^C$</td>
<td>$Y^P$</td>
<td>$Y^P$</td>
<td>$X^P$</td>
<td>$X^P$</td>
<td>$Z^I$</td>
<td></td>
</tr>
</tbody>
</table>

It has 10 Eckardt points. In addition to the 6 ideal Eckardt points, there are 4 non-ideal Eckardt points $e_0, e_1, e_2, e_3$ described as follows:

- $e_0 = (-2, -2, -2) = X^P_+ \cap Y^P_- \cap Z^P_-$
- $e_1 = (-2, 2, 2) = X^P_- \cap Y^P_- \cap Z^P_+$
- $e_2 = (2, -2, 2) = X^P_+ \cap Y^P_- \cap Z^P_+$
- $e_3 = (2, 2, -2) = X^P_+ \cap Y^P_- \cap Z^P_+$

If $p \in S_k$ which is the intersection of two lines $l_1, l_2 \subset S_k$ the intersection of $S_k$ with its tangent plane $T_p$ is a cubic curve in $T_p$ containing $l_1 \cup l_2$. Thus a third line $l_3$ exists with

$$S_k \cap T_p = l_1 \cup l_2 \cup l_3$$

and $T_p$ is a tritangent. In particular, for any Eckardt point $p$, the tangent plane $T_p$ is a tritangent.

In general the tritangents fall into two types: generic tritangent planes (which contain three lines in general position) and tangent planes at Eckardt points.
We relate the homogeneous coordinates $U_0, U_1, U_2, U_3, U_4$ to the our original homogeneous coordinates $X, Y, Z, W$:

\[
\begin{align*}
U_0 & \mapsto W \\
U_1 & \mapsto (-2W - X - Y + Z)/8 \\
U_2 & \mapsto (-2W - X + Y - Z)/8 \\
U_3 & \mapsto (-2W + X + Y - Z)/8 \\
U_4 & \mapsto (-2W + X + Y + Z)/8
\end{align*}
\]

(9)

The transposition $U_0 \leftrightarrow U_4$ corresponds to the involution

\[
(x, y, z) \mapsto (x, x + y + z - 2, x + y + z - 2, x + y + z - 2)
\]

which fixes the tritangent plane $x + y + z = 6$ (equal to $\mathcal{G}^{-1}_{\mathcal{T}_{e_0}}$). It maps the ideal tritangent plane $\mathcal{T}_{\infty}$ to the hyperplane $x + y + z = 2$, the plane containing the three Eckardt points $e_1, e_2, e_3$. The subgroup $\mathcal{S}_4 < \mathcal{S}_5$ consists of the linear automorphisms discussed earlier, and $\mathcal{S}_5 = (\mathcal{S}_4, (04))$.

6.1. **Dynamical significance.** The tritangent $\mathcal{T}_{e_0} = \mathcal{T}_{e_0} S_{18}$ defined by:

\[
x + y + z + (2 + \sqrt{k - 2}) = 0
\]

containing the three lines:

\[
\begin{align*}
c_{23} & = z^{P}_\infty := \{ z = -2, \quad x + y + \sqrt{k - 2} = 0 \} \\
c_{46} & = x^{P}_\infty := \{ x = -2, \quad y + z + \sqrt{k - 2} = 0 \} \\
c_{15} & = y^{P}_\infty := \{ y = -2, \quad z + x + \sqrt{k - 2} = 0 \}
\end{align*}
\]

is dynamically interesting, for $k \geq 18$. The orthant $\Omega$ defined by $x, y, z \leq -2$ parametrizes the Fricke space of the 3-holed sphere with the **standard marking**, the one whose generators correspond to the boundary components. It meets $S_k$ in a wandering domain for the action of

\[
\Gamma := \text{Out}(F_2) \cong \pi_0(\text{Homeo}(S)) \cong \text{GL}(2, \mathbb{Z})
\]

and is bounded by $T_{13}$. Geometrically, points in the orbit $\Gamma \Omega$ correspond to homotopy equivalences $S \to M$, where $M$ is a complete hyperbolic surface homeomorphic to a three-holed sphere.

The Eckardt point $e_0 := (-2, -2, -2)$ in the Clebsch cubic $S_{18}$ corresponds to the complete finite area 3-punctured sphere $M$. This Eckardt point arises as the domain $\Omega$ collapses as $k \searrow 18.$) The four $P$-lines in each coordinate family bound an open annulus, whose levels are ellipses. The corresponding cyclic group of Dehn twists acts minimally
(and ergodically) on almost every level ellipse. This leads to chaotic dynamics (ergodicity with respect to the Poisson measure arising from the invariant function $\kappa$ and Euclidean volume form) on the complement of the orbit $\Gamma \cdot \Omega$ of the wandering domain.

6.2. **Enumerating tritangents.** A general cubic surface contains 45 tritangent planes. We can account for them as follows on the Clebsch cubic $\mathcal{S}_{18}$, which has 4 finite Eckardt points.

First there is the ideal tritangent $T_\infty$ containing all three ideal lines. There are 30 tritangents arising from the double-six matrix. Namely the 12 s partition into the two sextuples $a_1, \ldots, a_6$ and $b_1, \ldots, b_6$. Whenever $i \neq j$, the lines $a_i, b_j$ extend to a tritangent also containing $c_{ij}$. The 15 lines $c_{ij}$ consist of all 12 of the P-lines and all 3 of the ideal lines. The ideal lines fall into the tritangents containing a pair of crossing C-lines.

Corresponding to 6 ideal Eckardt points (the common ideal points of a parallel pair of P-lines) are 6 tritangents.

Corresponding to each of the 4 finite Eckardt points is tritangent of concurrent P-lines. For example §6.1 discusses the Eckardt point

$$e_0 := c_{23} \cap c_{46} \cap c_{15} = ZP_- \cap XP_- \cap YP_-;$$

whose tangent plane $T_{e_0}S_{18}$ is the tritangent defined by

$$x + y + z + (2 + \sqrt{k-2}) = 0$$

Its image under the Galois involution $\mathcal{G}^-$ is another tritangent containing lines

$$ZP_- \cap XP_- \cap YP_-,$$

and defined by

$$x + y + z + (2 - \sqrt{k-2}) = 0.$$ 

Thus the 4 tangent planes to E-points and their Galois conjugates give a total of 8 tritangents.

This accounts for all $1 + 30 + 6 + 8 = 45$ tritangents to $\mathcal{S}_k$ in the special case $k = 18$.

7. **The Fermat surface**

This is the cubic surface in $\mathbb{CP}^3$

\begin{equation}
(U_0)^3 + (U_1)^3 + (U_2)^3 + (U_3)^3 = 0.
\end{equation}

The plane

\begin{equation}
U_0 + U_1 + U_2 + U_3 = 0,
\end{equation}
is a generic triangent plane $T$ and $S_{-10/3}$ is the corresponding affine cubic surface. $T$ meets the surface defined by (10) in the locus

$$(U_1 + U_2)(U_2 + U_3)(U_3 + U_1) = 0,$$

a union of three crossing lines, proving $T$ is generic, as claimed.

Idealizing (11), the resulting affine patch has affine coordinates $(u_1, u_2, u_3)$ with chart:

$$\mathbb{C}^3 \rightarrow \mathbb{C}P^3 \setminus T$$

$$(u_1, u_2, u_3) \mapsto \begin{bmatrix} 1 - u_1 - u_2 - u_3 \\ u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

so (10) becomes:

$$(1 - u_1 - u_2 - u_3)^3 + (u_1)^3 + (u_1)^3 + (u_2)^3 + (u_2)^3 = 0.$$

Apply the substitution

$$x := 4(u_2 + u_3) - 2$$
$$y := 4(u_3 + u_1) - 2$$
$$z := 4(u_2 + u_3) - 2$$

to obtain:

$$(1 - u_1 - u_2 - u_3)^3 + (u_1)^3 + (u_1)^3 + (u_2)^3 + (u_2)^3 = \frac{3 \kappa(x, y, z) + 10}{64}.$$ 

Therefore this affine piece of the Fermat surface identifies with $S_{-10/3}$.

8. **Sylvester pentahedral form**

This can be cast in a more general context using the *Sylvester pentahedron*. Namely the substitution (9) maps the projective variety with homogeneous coordinates $U_0, U_1, U_2, U_3, U_4$ to the the original homogeneous coordinates $X, Y, Z, W$. In particular (9) maps the homogeneous equation

$$\frac{3k + 10}{64}(U_0)^3 + (U_1)^3 + (U_2)^3 + (U_3)^3 + (U_4)^3 = 0,$$

to the inhomogeneous equation

$$x^2 + y^2 + z^2 - xyz - 2 = k$$

where $x = X/W, y = Y/W, z = Z/W$. The *Sylvester pentahedron* is the configuration formed by the hyperplanes defined by the $U_i$. 
In particular the case $k = -10/3$ corresponds to the case when the coefficient of the $U_0$-term vanishes (the Fermat surface), the case $k = -2$ (the Markoff surface) corresponds to
\[ \frac{1}{8} (U_0)^3 + (U_1)^3 + (U_2)^3 + (U_3)^3 + (U_4)^3 = 0, \]
the case $k = 2$ (the Cayley surface) corresponds to
\[ \frac{1}{2} (U_0)^3 + (U_1)^3 + (U_2)^3 + (U_3)^3 + (U_4)^3 = 0, \]
and the Clebsch surface ($k = 18$) corresponds to
\[ (U_0)^3 + (U_1)^3 + (U_2)^3 + (U_3)^3 + (U_4)^3 = 0, \]
the surface with full $S_5$-symmetry.

Notation

Ideal lines: $X\mathcal{I}$, $Y\mathcal{I}$, $Z\mathcal{I}$
Coordinate ideal points: $X\mathbb{P}_\infty$, $Y\mathbb{P}_\infty$, $Z\mathbb{P}_\infty$
$C$-lines: $X\mathcal{C}^\pm$, $Y\mathcal{C}^\pm$, $Z\mathcal{C}^\pm$
$P$-lines: $X\mathcal{P}^\pm$, $Y\mathcal{P}^\pm$, $Z\mathcal{P}^\pm$
Ideal tritangent plane: $\mathcal{T}_\infty$
Ideal Eckardt points: $X\text{Eck}^\pm$, $Y\text{Eck}^\pm$, $Z\text{Eck}^\pm$
Finite Eckardt points: $e_0, e_1, e_2, e_3$
Symmetric groups: $\mathfrak{S}_3, \mathfrak{S}_4, \mathfrak{S}_5$
Critical points of $\kappa$: $c_0, c_1, c_2, c_3$
Dihedral characters: $X\text{Dih}^\pm$, $Y\text{Dih}^\pm$, $Z\text{Dih}^\pm$
Sign-changes $\sigma_1, \sigma_2, \sigma_3$ comprise $\Delta$ realizing $(\mathbb{Z}/2 \oplus \mathbb{Z}/2) \triangleleft \mathfrak{S}_4$
Vieta involutions $X\nu$, $Y\nu$, $Z\nu$

References

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