PROPER ACTIONS IN PSEUDO-RIEMANNIAN GEOMETRY

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Contents

Introduction

Toshiyuki Kobayashi has made fundamental contributions to the study of proper transformation groups. Informally speaking, *properness* is a uniformity condition on a group action which ensures good behavior of the quotient space. When the group is equipped with the discrete topology, this is the familiar notion of a *properly discontinuous* action:

> A group action $G \curvearrowright X$ is *properly discontinous* if for compact subsets $K_1, K_2 \subset X$, for only finitely many $g \in G$ is $K_1 \cap gK_2$ nonempty.

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When the action is *free,* then this implies that the quotient map $X \to G\backslash X$ is a covering space. However, even if the G enjoys the discrete topology as a subgroup of the homeomorphism group $\mathsf{Homeo}(X)$, it need not act properly discontinuously.

Much of Kobayashi's work concerns groups of isometries of pseudo-Riemannian structures, in which case the quotient $G\backslash X$ itself has a pseudo-Riemannian structure for whic $X \to G\backslash X$ is a local isometry. A more extensive survey of this work can be found in Constantine [8] and Kobayashi's papers [32, 34] to which we refer for details. We describe how some of the original insights of Kobayashi in this subject have led to major developments in differential geometry and Lie theory, having deep inter-relationships with other fields of mathematics.

1. Homogeneous spaces and biquotients

Biquotients $\Gamma\backslash G/K$, where G is a Lie group, $\Gamma < G$ is a discrete subgroup and $K < G$ is a compact subgroup, arise as fundamental objects in many mathematical contexts. When Γ is torsionfree, and K is the maximal compact subgroup, then $\Gamma\backslash G/K$ is a manifold with a *locally symmetric* complete Riemannian metric. In his seminal paper [5], A. Borel showed that every semisimple Lie group G admits such a *compact* biquotient, which is a called a *compact Clifford-KIein form.* This is equivalent to $\Gamma < G$ being a cocompact (or *uniform*) lattice.

It is natural to ask whether such manifolds exist when K is replaced by a closed subgroup $H < G$, which is not necessarily compact. A basic example occurs with *de Sitter space,* which can be defined as the quotient

$$\mathsf{dS}^3_1 := \mathsf{O}(2,1)/\mathsf{O}(1,1).$$

(That is, $G = \mathsf{O}(2,1)$ and $H = \mathsf{O}(1,1)$.) This identifies with the *one-sheeted hyperboloid* $x^2+y^2-z^2 = 1$, where the defining quadratic form for $\mathbb{R}^{2,1}$ is

$$Q(x,y,z) = x^2 + y^2 - z^2,$$

and the stabilizer of $(1,0,0)$ identifies with $\mathsf{O}(1,1)$. Calabi and Markus [7] showed that no if $\Gamma < G$ is a subgroup such that $\Gamma\backslash G/H$ is a manifold, then Γ must be finite.

This markedly contrasts the hyperbolic surfaces $\Gamma\backslash G/K$. The maximal compact subgroup K of $\mathsf{O}(2,1)$ equals $\mathsf{O}(2)\times\mathsf{O}(1)$, the symmetric space G/K identifies with the hyperbolic plane (one component of the two-sheeted hyperboloid defined by $x^2+y^2-z^2 = -1$). Then many discrete subgroups $\Gamma < G$ exist, for example, representing all hyperbolic surfaces, including closed surfaces of genus > 1.

2. Proper actions

In this section we briefly review the theory of *proper* actions, to describe conditions which provide Hausdorff quotient space of discrete group actions.

2.1. **Proper actions, fibrations, and covering spaces.** Let X, Y be locally compact Hausdorff spaces. We use the notation that $A \subset\subset B$ means "A is a compact subset of B." Recall that a continuous map $X \xrightarrow{f} Y$ is *proper* if $\forall K \subset\subset Y$, the preimage $f^{-1}K \subset\subset X$. If G is a locally compact topological group acting on X, then the action is said to be *proper* if and only if the corresponding mapping

$$\begin{aligned} G \times X &\longrightarrow X \times X \\ (g, x) &\longmapsto (g \cdot x, x) \end{aligned}$$

is proper. What this means is that "going to infinity" in the orbits of G implies "going to infinity (uniformly) in G." This global uniformity condition has many consequences:

- The quotient $G\backslash X$ is Hausdorff;
- Every orbit is closed;
- G is closed in the group $\mathsf{Homeo}(X)$ comprising homeomorphisms $X \to X$, given the compact-opern topology;
- If G acts freely, the quotient map $X \longrightarrow G\backslash X$ is a fibration.

This notion was introduced by Palais [41] to study group actions whose quotients have good local behavior. In particular he investigates, when the action possesses *slices,* that is, cross-sections $S \hookrightarrow X$ which are left-inverses to $X \to G\backslash X$: points of S represent uniquely the G-orbits in X.

When G is discrete, this is the usual notion of "properly discontinuity," and the quotient map is a branched covering space (a covering space if the action is free). The terminology has evolved somewhat unfortunately and errors in the literature abound. Sometimes "properly discontinuous" is called "totally discontinuous." Sometimes the term "discontinuous" is used for the weaker notion of *wandering:* An action of G on X is said to be wandering if every point $x \in X$ has an open neighborhood U such that the set of $g \in G$ for which $g(U) \cap U \neq \emptyset$ is finite. This condition is not strong enough to imply that the quotient X/G is Hausdorff. Indeed, Wolf [48, 47] uses the term "properly discontinuous" when he really means "wandering". Sometimes "wandering" is called "weakly proper," or more suggestively, *locally proper.* Kulkarni [35] suggests that "properly discontinuous" really should be "discretely proper" since it refers to a proper action when G is given the discrete topology. (This also avoids the term "discontinuous" which I find confusing since all the mappings involved are continuous mappings.) While the quotient by a proper action is Hausdorff (T^2), the quotient by a wandering action is only T^1 — points in $G\backslash X$ are closed subsets, but may not have disjoint open neighborhoods. This phenomenon appears for the basic example in §2.2.

It is easy to see that a transitive action of G on X is proper $\Longleftrightarrow$ the *stablizer subgroup*

$$\mathsf{Stab}(p) := \{g \in G \mid g(p) = p\}$$

is compact for some (and hence every) $p \in X$.

2.2. **The basic example.** Here is a basic example of a linear action of the additive group $\mathbb{R}$ which is not proper. In a certain sense, it arises for any non-proper action of $\mathbb{R}$.

Namely, consider the partially closed quadrant

$$W := \{(x, y) \in \mathbb{R}^2 \mid x, y \geq 0\} \setminus \{(0, 0)\},$$

its two boundary components

$$X := \mathbb{R}_+ \times \{0\}, \quad Y := \{0\} \times \quad \mathbb{R}_+$$

and its interior

$$\mathsf{int}(W) = \mathbb{R}_+ \times \mathbb{R}_+.$$

The one-parameter group

$$G = \left\{ g_t := \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}, t \in \mathbb{R} \right\}.$$

acts freely on W, X, Y and $\mathsf{int}(W)$. Furthermore G acts properly on $X, Y, \mathsf{int}(W)$ but not on W.

To see this, consider $\epsilon > 0$ and the compact subsets of W:

$$X_\epsilon := \{1\} \times [0, \epsilon]$$
$$Y_\epsilon := [0, \epsilon] \times \{1\}$$

Then $(1, \epsilon) \xrightarrow{g_t} (\epsilon, 1) \Longleftrightarrow t = \log(\epsilon)$. As $\epsilon \to 0$, the time interval $t \nearrow +\infty$. In particular

$$K := X_\epsilon \cup Y_\epsilon \subset\subset W$$

but

$$\{g \in G \mid gK \cap K \neq \emptyset\}$$

is a noncompact subset of G. In the quotient W/G, points on the image of X and the image of Y cannot be separated by open sets. Compare Figure 1.

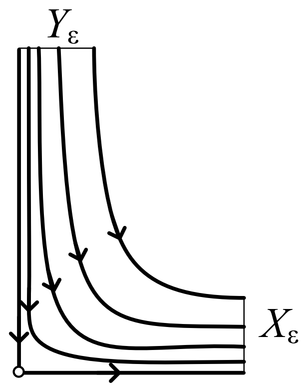

FIGURE 1. A basic example of a non-proper linear action.

This general picture appears in any non-proper group action. Namely, if G is a group acting on a space X and the action is *not* proper, then sequences $x_0, x_1, x_2, x_3, \cdots \in X$ and $g_1, g_2, g_3, \cdots \in G$ exist, with

$$g_n(x_0) = x_n$$

such that

$$\lim_{n\to\infty} x_n =: x_\infty$$

exists, but the sequence g_n diverges. In the case of a flow, this is the above picture, where it takes longer and longer to get from near x_∞ to near x_0. Conversely if the action is proper, for every divergent sequence $g_n \in G$ and $x_0 \in X$, the sequence $g_n(x_0) \in X$ diverges.

This basic example is used in Goldman-Labourie-Margulis [18] to show that an affine deformation of a hyperbolic surface Σ gives rise to a function on probability measures invariant under the geodesic flow of Σ. The positivity of this function is equivalent to the properness of the deformation. In particular, the presence of this basic example gives a probability measure on which this function vanishes. Compare also [9] and the references given there.

3. Pseudo-Riemannian symmetric spaces

A classic theorem in differential geometry is that a compact Riemannian manifold (M, g) is geodesically complete. If (M, g) is locally homogeneous, then M is the quotient of a model space X by a discrete subgroup $\Gamma < \mathsf{Isom}(X)$, which necessarily acts properly.

However, the extension of this circle of ideas to the case when g is indefinite is highly nontrivial, interesting and still mysterious.

3.1. **Calabi-Markus phenomenon.** Calabi and Markus [7] showed that a complete de Sitter manifold cannot be compact. Specifically, they show every discrete subgroup of $\mathsf{SO}(p+1,1)$ acting properly on $\mathsf{dS}^k = \mathsf{SO}(k+1,1)/\mathsf{SO}(p,1)$ *must be finite.*.

The proof can be understand by the following simple idea. Suppose that $X = \mathsf{dS}^2$ be 2-dimensional de Sitter space; as a homogenous space it identifies with $\mathsf{SO}(2,1)/\mathsf{SO}(1,1)$, and $X \approx \mathsf{S}^1 \times \mathbb{R}$. In projective space it identifies with the quadric

$$-X^2 - Y^2 + 4(Z^2 + W^2) = 0$$

in homogeneous coordinates which in turn identifies with the *hyperbolic paraboloid* (or *saddle*) defined by $x = y^2 - z^2$ in $\mathbb{R}^3$ arising from the affine chart

$$\mathbb{R}^3 \hookrightarrow \mathsf{P}^3$$

$$(x,y,z) \longmapsto \begin{bmatrix}\begin{bmatrix} x-1 \\ y \\ z \\ x+1 \end{bmatrix}\end{bmatrix}$$

The flow

$$(x,y,z) \xmapsto{\phi_t} \big(x, \cosh(t)y + \sinh(t)z, \sinh(t)y + \cosh(t)z\big)$$

extends to a one-parameter subgroup of orthogonal matrices, preserves the saddle, and may be seen more vividly using a simple change of linear coordinates $(y,z) = \big((u+v)/2, (v-u)/2\big)$ (where $u = y - z$ and $v = y + z$)):

$$(u,v) \longmapsto (e^{-t}u, e^t v),$$

the basic example of a non-proper $\mathbb{R}$-action. This is the simplest example of the Calabi-Markus phenomenon. (Compare Figure 2.)

If $M = \Gamma\backslash X$ is a complete de Sitter manifold, then Γ must necessarily be infinite and discrete. Thus $\exists \gamma \in \Gamma$ generating a discrete cyclic subgroup $\langle\gamma\rangle$ which acts properly on X.

3.2. **Compact pseudo-Riemannian space forms.** When $G/H = \mathsf{SO}(p+1,q)/\mathsf{SO}(p,q)$, then G admits an infinite discrete group Γ acting properly on $G/H \iff p < q$ (the Calabi-Markus phenomenon). Furthermore if Γ can be chosen so that $\Gamma\backslash G/H$ is compact, and $pq > 0$, then q must be even (we take $p < q$). He found explicit examples for all $\mathsf{SO}(1,2n)$, $\mathsf{SO}(3,4n)$ and $\mathsf{SO}(7,8)$, Kobayashi has conjectured that these are the only possible values of (p,q).

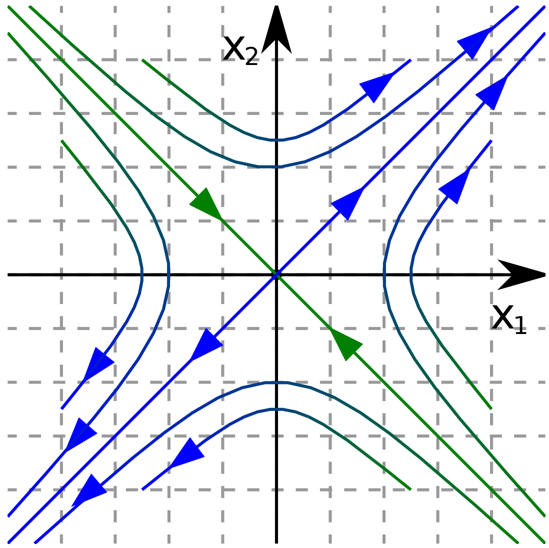

FIGURE 2. An improper saddle point arising from an isometric flow on dS^2.

3.3. **Nilpotent groups: Lipsman's conjecture.** Opposite from reductive groups are *nilpotent groups,* where analogous questions were studied by Ron Lipsman [37], who proposed the following conjecture:

> If G is a 1-connected nilpotent Lie group, where $L \subset G \supset H$ are closed subgroups, then L acts freely on $X = G/H$ $\Longrightarrow$ L acts properly on X.

Lipsman's paper was inspired by Kobayashi's paper [27], which he cites as a "basic paper", in which Kobayashi discovered the strong parallels between the problem of proving the quotient X/L is a manifold, and passing from L to its Zariski closure in G (when G is an algebraic group). This latter construction is especially strong when G is a 1-connected nilpotent Lie group, which inherits an algebraic structure from a faithful unipotent representation. In this case, the Zariski closure of a discrete subgroup L is the *Malcev completion,* which can be carried out purely in terms of the group theory of L (compare the brief discussion in [16], §8.6).

The *compact intersection property,* from [27] is the statement that each $\mathsf{Stab}(p) \subset\subset G$, and is equivalent to properness for transitive actions. The triviality of compact unipotent groups provides evidence for Lipsman's conjecture.

Nasrin [38] proved Lipsman's conjecture if G is 2-step nilpotent, and Baklouti [2] and Nasrin in 2007 extended this for 3-step nilpotent groups. However, in 2005 Yoshino [49] disproved Lipsman's conjecture with a 4-step nilpotent group G containing $L \cong \mathbb{R}^2$ and $X \approx \mathbb{R}^2$ a nilmanifold quotient of G such that L:

- L acts freely on X;
- All L-orbits are closed;
- $L\backslash X$ is not Hausdorff.

Kobayashi and Nasrin [33] completely determine the deformation space of proper actions of the free abelian group $\mathbb{Z}^k$ on a $k+1$-dimensional affine space by a $2k+1$-dimensional 2-step nilpotent group (the *Heisenberg group*). They prove the suggestive fact that every such proper action extends to an action of $\mathbb{R}^k \supset \mathbb{Z}^k$, and conclude with explicitly showing the space of such proper actions contains on open subset of dimension 2 if:

$$\begin{cases} 2 & \text{if } k = 1 \\ 2k^2 - 2 & \text{if } k > 1 \text{ is odd} \\ 2k^2 - 1 & \text{if } k > 1 \text{ is even} \end{cases}$$

However, the situation is much more subtle and complicated when the groups are *reductive,* involving the development of new machinery. This sheds light on the extension of the Calabi-Markus phenomenon and the classification of Clifford-Klein forms for reductive homogeneous spaces.

4. Properness criteria

The asymptotics of a semisimple Lie group are governed by Cartan's $K\mathsf{A}K$-decomposition, where $K < G$ is a maximal compact and A is an $\mathbb{R}$-split torus: going to ∞ in G means going to ∞ in A. Using this paradigm, Kobayashi [30] found an elegant properness criterion, which was obtained independently by Yves Benoist [3],§3.1 about the same time. This criterion relies on Kobayashi's idea in the earlier properness criterion from [26].

4.1. **Properness and similarity.** We assume that G is a locally compact group, and L and H are subsets. If $S \subset\subset G$ and $U \subset G$, think of the product

$$SUS := \{ s_1 u s_2 \mid s_1, s_2 \in S, u \in U \}$$

as a "compact thickening" of an arbitrary subset $U \subset G$, which will be closed if U is closed, and will be compact if $U \subset\subset G$.

Say that $L \pitchfork H$ if and only if $\forall S \subset\subset G$, the closure

$$\overline{L \cap SHS} \subset\subset G,$$

and $L \sim H$ (L and H are *similar* in G) if and only if $\exists S \subset\subset G$ such that

$$L \subset SHS \text{ and } H \subset SLS.$$

Clearly similarity is an equivalence relation. Furthermore $\sim$ and $\pitchfork$ are related by:

$$L \sim H \iff \forall M \subset G, L \pitchfork M \iff H \pitchfork M$$

For example, in the basic case when G is a vector space $\mathbb{R}^n$ and $L, H < G$ are vector subspaces, then $L \pitchfork H \iff L \cap H = \{0\}$ and $L \sim H \iff L = H$.

The relations $\pitchfork$ and $\sim$ are especially useful because, in the reductive case, they behave well in reducing to the maximal split torus A. Let

$$G \xrightarrow{\nu} \mathsf{A}$$
$$k_1 a k_2 \longmapsto a$$

be the retraction (Cartan projection) $G \to \mathsf{A}$. That these relations are valid for subsets which are not necessarily subgroups (such as the Cartan projections $\nu(L)$ provides extra flexibility letting them to be a powerful tool in analyzing proper group actions:

Proposition 4.1. *(Benoist [3], Kobayashi [30]) If $K, L < G$, then*

- *$H \sim L$ in $G \iff \nu(H) \sim \nu(L)$ in A;*
- *$H \pitchfork L$ in $G \iff \nu(H) \pitchfork \nu(L)$ in A.*

This A-reduction is a key tool in the extensions of Calabi-Markus [7] by Wolf [48, 47] Kulkarni [35] and Kobayashi [26]:

Proposition 4.2. *Suppose $H < G$ are real reductive Lie groups of respective $\mathbb{R}$-ranks $\ell(H) \leq \ell(H)$. Then $\ell(H) = \ell(G)$ if and only if $\nexists \Gamma < G$ which is infinite and acts properly on G/H.*

In a similar direction, Benoist [3] proved that, for a reductive homogeneous space G/H, a discrete two-generator free subgroup $\Gamma < G$ exists which acts properly on G/H only if $\ell(G) > \ell(H)$.

4.2. **Compact Clifford-Klein forms.** A natural question is whether for a given homogeneous space G/H a discrete subgroup $\Gamma < G$ exists for which the action is proper, and the quotient $\Gamma \backslash G/H$ is *compact.* This is an extremely difficult and intricate question upon Kobayashi has worked. This question brings in many tools from group theory and topology.

Some of the earliest results follow from the techniques described above. For example, Benoist [3] proved that $\mathsf{SL}(2n+1, \mathbb{R})/\mathsf{SL}(2n, \mathbb{R})$ has no compact Clifford-Klein form, and Oh-Witte [39] proved that for $G = \mathsf{SL}(3,\mathbb{R})$, then G/H has a compact quotient unless either H or G/H is compact. (Compare also Iozzi-Witte [17].) In another paper [40], they conjecture that $\mathsf{SO}(2,2m+1)/\mathsf{SU}(1,m)$ has no compact Clifford-Klein forms. This implies that $\mathsf{SO}(2,2m+1)$ has no interesting compact biquotients whatsoever. They give many examples for $\mathsf{SO}(2,2m)$, including the observation of Kulkarni [35] that even-dimensional compact $\mathbb{R}$-hyperbolic manifolds give rise to compact Clifford-Klein forms of $\mathsf{SO}(2,2m)/H$ for many closed subgroups $H < \mathsf{SO}(2,2m)$.

Of course Propostion 4.2 gives necessary conditions but gives no information on compactness. Given a reductive group G, define the *noncompactness dimension* $d(G)$ as the dimension of G/K, where $K \subset\subset G$ is a maximal compact subgroup. In particular if $\Gamma < G$ is a discrete torsionfree subgroup, then its *cohomological dimension* $\mathsf{cd}(\Gamma)$ satisfies:

$$\mathsf{cd}(\Gamma) \leq \mathsf{dim}(\Gamma\backslash G/K) = \mathsf{dim}(G/K) =: d(G)$$

with equality holding if $\Gamma\backslash G/K$ is compact.

The following theorem is proved in Kobayashi [26]:

Theorem 4.3. *Let G/H be a reductive homogeneous space. If $\exists H' < G$ such that $H \sim H'$ and $d(H') > d(H)$, then G/H admits no compact Clifford-Klein form.*

This generalizes Kulkarni's generalization [35] of Calabi-Markus [7] for $\mathsf{SO}(p+1,q)/\mathsf{SO}(p,q)$. See Constantine [8] for more information on this fascinating question.

4.3. **Standard quotients.** If G acts *properly* on X, then obviously any closed subgroup (such a discrete subgroup) also acts properly on X. For example, the group of isometries of a complete Riemannian manifold (X,g) acts properly on X, so every discrete subgroup Γ of $\mathsf{Isom}(X,\mathsf{g})$ acts properly on X. If (X,g) is a *homogeneous Riemannian manifold,* (that is, $\mathsf{Isom}(X,\mathsf{g})$ acts transitively on X), then the quotient $\Gamma\backslash X$ is a Clifford-Klein form of X.

However, for most homogeneous pseudo-Riemannian manifolds (X,g), the transitive action of $\mathsf{Isom}(X,\mathsf{g})$ is *not* proper. Here is a general approach for constructing Clifford-Klein forms of X.

Suppose $\mathsf{Isom}(X,\mathsf{g})$ contains a closed subgroup G which does act transitively and properly on X. Such a quotient $\Gamma\backslash X$, where

$$\Gamma < G < \mathsf{Isom}(X,\mathsf{g})$$

is a discrete subgroup, is called a *standard* quotient.

A basic question in this direction was conjectured by Kobayashi [26]:

> A reductive homogeneous space G/H admits a compact Clifford-Klein form $\Longleftrightarrow G/H$ admits a compact *standard* Clifford-Klein form.

The first examples of compact Lorentzian space forms of nonzero curvature are due to Kulkarni and Raymond[36]. These are anti-de Sitter 3-manifolds which admit timelike Killing vector fields. One obtains a Riemannian structure by combining the anti-de Sitter Lorentzian structure with the timelike Killing vector field to form a homogeneous Riemannian manifold. This homogeneous Riemannian manifold is one of Thurston's eight geometries[46] coming from a left-invariant Riemannian structure on $\mathsf{SL}(2,\mathbb{R})$.

4.4. **Nonstandard examples.** The first examples of *nonstandard* Clifford-Klein forms are due to the author [15] and Ghys [14], obtained by continuously deforming standard structures. Later Salein [43] found further examples which are not continuous deformations of standard structures.

Anti de Sitter space AdS^3 has an alternate and suggestive description. The Lie group $G = \mathsf{SL}(2,\mathbb{R})$ has a bi-invariant Lorentzian metric arising from the Killing form on its Lie algebra, The action of $G \times G$ by left- and right- multiplications is isometric with respect to this Lorentzian metric, and defines a model for AdS^3, where $\mathsf{AdS}^3 \leftrightarrow G$ and $\mathsf{Isom}(\mathsf{AdS}^3) \leftrightarrow G \times G$ (up to *local* isomorphisms). Kulkarni and Raymond [36] prove that if Γ is a discrete group of isometries acting properly, then Γ corresponds (up to interchanging the factors) the graph of a homomorphism $\Gamma \xrightarrow{\rho} G$. (Compare also Dumtrescu-Zeghib [12] and Zeghib [50].) That is,

$$\Gamma = \mathsf{graph}(\rho) := \{\big(\gamma, \rho(\gamma)\big) \mid \gamma \in \Gamma\}$$

where the projection on the first factor of $G \times G$ is an embedding $\Gamma \hookrightarrow G$ onto a discrete subgroup of G. The standard examples they construct correspond to the case when $\rho(\Gamma) < \mathsf{SO}(2) < G$. This is equivalent to the AdS-manifold to possess a timelike Killing vector field.

For the nonstandard examples in [15], ρ takes values in a hyperbolic one-parameter subgroup $\mathsf{A} < G$. A key point is that these deformations are *geodesically complete,* which follows from a general theorem of Klingler [23] on completeness of closed Lorentz space forms. (Compare [16], §8, for a general discussion of completeness of geometric structures.

In [15], I conjectured that the graph $\mathsf{graph}(\rho)$ acts properly whenever ρ is sufficiently near the trivial representation. This conjecture was proved in Kobayashi [31]. Furthermore he constructs similar examples in $\mathsf{SO}(2n,2)/\mathsf{SO}(2n,1)$, $\mathsf{SU}(2n,2)/\mathsf{Sp}(n,1)$, $\mathsf{SO}(4,3)/\mathsf{G}_2(\mathbb{R})$ and $\mathsf{SO}(4,4)/\mathsf{Spin}(4,3)$.

Related results from Kobayashi [29] and [31] concern generalizing the graph construction from $\mathsf{SL}(2,\mathbb{R})$ above to other groups. In particular he also shows that the only examples of other simple Lie groups G for which the construction with $G \times G$ is possible occur for G isomorphic to $\mathsf{SO}(n,1)$ or $\mathsf{SU}(n,1)$.

AdS^3-manifolds provide a wealth of new phenomena, which have been the focus of much recent activity. The graph construction and the properties of surface group representations leads to many open questions, relating the classical Clifford-Klein problem to what has been known as "higher rank Teichmüller theory" [6]. These ideas intimately relate to Kobayashi's work. Among some of these recent develpments are Danciger-Guéritaud-Kassel [10], Deroin-Tholozan [11], Tholozan [44, 45], Gueritaud-Guichard-Kassel-Wienhard [19, 20], Gueritaud-Guichard-Kassel-Wienhard-Wolff [21], Kapovich-Leeb-Porti [24]. In particular properness of $\mathsf{graph}(\rho)$ corresponds to the representation ρ having the property that it shortens all the simple geodesics on the hyperbolic surface $\Gamma\backslash\mathsf{H}^2$. Compare also the brief discussion in [16], §15.3–5.

4.5. **New directions.** The recent paper of Kobayashi-Yoshino [34] contains several striking results which deserve mention, indicating the diversity of mathematical subjects related to this subject.

Recall that a symmetric space is a homogeneous space $M = G/H$ where H is a closed subgroup of the Lie group G and for which there exists an automorphism $\sigma \in \mathsf{Aut}(G)$ of order two such that $G\sigma \subset G \subset G\sigma$, where $G\sigma$ is the identity component of the closed subgroup $G\sigma$. If $\Gamma$is a discrete subgroup of G, which acts properly discontinuously and freely on the symmetric space G/H, then the double coset space $\Gamma\backslash G/H$ naturally inherits a manifold structure, and such a manifold is said to be a Clifford-Klein form of G.

A particularly interesting example arises from the *complex sphere:* the n-dimensional quadric $z_0^2 + \dots z_n^2 = 1$ in $\mathbb{C}^n$, identifying with the homogeneous space $\mathsf{SO}(n+1,\mathbb{C})/\mathsf{SO}(n,\mathbb{C})$. Kobayashi and Yoshino [34] proves that this symmetric space admits a compact Clifford-Klein form.

Another result I find particularly striking concerns *tangential symmetric spaces,* a generalization of Euclidean geometry. Associated to a symmetric space G/H is another symmetric space, its *tangential symmetric space* G_θ/H_θ, where θ is the involution fixing the Lie subalgebra

$\mathfrak{h}$ corresponding to H. The involution determines an eigenspace decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ and the tangential symmetric space is defined by:

$$G_\theta := G \ltimes \mathfrak{p}, H_\theta := G$$

Kobayashi and Yoshino [34] characterize when the tangential symmetric space associated to $\mathsf{SO}(p, q+1)/\mathsf{SO}(p, q)$ admits compact Clifford-Klein form. They relate it to the *Hurwitz-Radon number* which arises in the famous theorem of J. F. Adams [1] on the largest rank of a trivial subbundle of the tangent bundle to the Euclidean sphere, as well as problems in linear algebra on the factorization of quadratic forms (Radon [42], Hurwitz [22], Eckmann [13]).

Finally, relating the topology and deformation theory of Clifford-Klein forms to the function theory provides finer quantitative information and more refined notions of proper actions. This ambitious program has been started by Kassel and Kobayashi [25], and more recent work by Benoist and Kobayashi [4].

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