ZBLMATH REVIEW OF "RANDOM UNITARY REPRESENTATIONS OF SURFACE GROUPS I: ASYMPTOTIC EXPANSIONS" BY MICHAEL MAGEE

WILLIAM M. GOLDMAN

In this very interesting paper, the author studies the moduli space of representations of the fundamental group Γ_g of a closed orientable surface of genus $g > 1$ into the special unitary group $\mathsf{SU}(n)$ where $n > 1$. In particular he considers random representations, with respect to a finite measure defined by the symplectic structure on the quotient

$$\mathfrak{X}_{g,n} := \mathsf{Hom}(\Gamma_g), \mathsf{SU}(n)\big)/\mathsf{Inn}\big(\mathsf{SU}(n)\big),$$

studied by Narasimhan, Atiyah, Bott, the reviewer, Huebschmann and others. The paper concerns the asymptotic behavior as $n \longrightarrow \infty$.

The symplectic structure arises from combining the orientation of Σ_g and an Ad-invariant inner product B_n on the Lie algebra $\mathfrak{su}(n)$. The bilinear form B_n is necessarily a multiple of the Killing form. The inner product B_n is normalized so that the corresponding bi-invariant Riemannian structure on $\mathsf{SU}(n)$ has Riemannian volume 1. With this normalization the total volume of $\mathfrak{X}_{g,n}$ equals $n\zeta(2g-2, n)$ where $\zeta(2g-2, n)$ refers to the the *Witten ζ-function*:

$$\zeta(s, n) := \sum_{(\rho,W)\in\widehat{\mathsf{SU}(n)}} \frac{1}{(\mathsf{dim} W)^s},$$

which was found by Witten and later rigorously proved by Sengupta.

The moduli space $\mathfrak{X}_{g,n}$ carries natural functions given by the characters of representations applied to elements of Γ_g, and is sometimes called the *character variety.* For $\gamma \in \Gamma_g$, the expected value $E_{g,n}(\gamma)$ of the function tr_γ on $\mathfrak{X}_{g,n}$ is bounded above in absolute value by n, and this bound is attained when $\gamma = e \in \Gamma_g$. For other elements $\gamma \in \Gamma_g$, conjecturally

$$\lim_{n\to\infty} \frac{E_{g,n}(\mathsf{tr}_\gamma)}{n} = 0,$$

and one corollary of the main result of this paper is the existence of this limit.

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The main result is an asymptotic formula for this expected value in terms of a sequence $a_{-1}(\gamma), a_0(\gamma), a_1(\gamma), \cdots + a_M(\gamma) \in \mathbb{Q}$, for any $M \in \mathbb{N}$:

$$E_{g,n}(\mathsf{tr}_\gamma) = a_{-1}(\gamma)n + a_0(\gamma) + \frac{a_1(\gamma)}{n} + \dots \frac{a_{M-1}(\gamma)}{n^{M-1}} + O(n^{-M})$$

For example, when γ corresponds to a separating simple closed curve on Σ_2, it is proved that

$$E_{g,n}(\mathsf{tr}_\gamma) = \frac{2}{n} + \frac{5}{n^3} + O(n^{-5}).$$

More asymptotic formulas are proved in this paper in terms of the Witten zeta-function and unitary dual of $\mathsf{SU}(n)$. Every irreducible $\mathsf{SU}(n)$-representation contributes to the expected values. The calculations heavily use all the machinery of representation theory of $\mathsf{SU}(n)$. Part of their technical difficulty is due to the growth of representations of $\mathsf{SU}(n)$ as $n \nearrow +\infty$. The paper also describes closely related questions in mathematical physics, notably expected values of Wilson loops in 2D Yang-Mills theory.

Department of Mathematics, University of Maryland, College Park, MD 20742, USA

Email address: wmg@umd.edu