ZBLMATH REVIEW OF “RANDOM UNITARY REPRESENTATIONS OF SURFACE GROUPS I: ASYMPTOTIC EXPANSIONS” BY MICHAEL MAGEE

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In this very interesting paper, the author studies the moduli space of representations of the fundamental group \( \Gamma_g \) of a closed orientable surface of genus \( g > 1 \) into the special unitary group \( SU(n) \) where \( n > 1 \). In particular he considers random representations, with respect to a finite measure defined by the symplectic structure on the quotient

\[
\mathfrak{X}_{g,n} := \text{Hom}(\Gamma_g, SU(n))/\text{Inn}(SU(n)),
\]

studied by Narasimhan, Atiyah, Bott, the reviewer, Huebschmann and others. The paper concerns the asymptotic behavior as \( n \to \infty \).

The symplectic structure arises from combining the orientation of \( \Sigma_g \) and an \( \text{Ad} \)-invariant inner product \( B_n \) on the Lie algebra \( su(n) \). The bilinear form \( B_n \) is necessarily a multiple of the Killing form. The inner product \( B_n \) is normalized so that the corresponding bi-invariant Riemannian structure on \( SU(n) \) has Riemannian volume 1. With this normalization the total volume of \( \mathfrak{X}_{g,n} \) equals \( n\zeta(2g-2,n) \) where \( \zeta(2g-2,n) \) refers to the the \textit{Witten} \( \zeta \)-function:

\[
\zeta(s,n) := \sum_{(\rho,W) \in SU(n)} \frac{1}{(\dim W)^s},
\]

which was found by Witten and later rigorously proved by Sengupta.

The moduli space \( \mathfrak{X}_{g,n} \) carries natural functions given by the characters of representations applied to elements of \( \Gamma_g \), and is sometimes called the \textit{character variety}. For \( \gamma \in \Gamma_g \), the expected value \( E_{g,n}(\gamma) \) of the function \( \text{tr}_\gamma \) on \( \mathfrak{X}_{g,n} \) is bounded above in absolute value by \( n \), and this bound is attained when \( \gamma = e \in \Gamma_g \). For other elements \( \gamma \in \Gamma_g \), conjecturally

\[
\lim_{n \to \infty} \frac{E_{g,n}(\text{tr}_\gamma)}{n} = 0,
\]

and one corollary of the main result of this paper is the existence of this limit.

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The main result is an asymptotic formula for this expected value in terms of a sequence $a_{-1}(\gamma), a_0(\gamma), a_1(\gamma), \cdots + a_M(\gamma) \in \mathbb{Q}$, for any $M \in \mathbb{N}$:

$$E_{g,n}(\text{tr} \gamma) = a_{-1}(\gamma)n + a_0(\gamma) + \frac{a_1(\gamma)}{n} + \cdots + \frac{a_{M-1}(\gamma)}{n^{M-1}} + O(n^{-M})$$

For example, when $\gamma$ corresponds to a separating simple closed curve on $\Sigma_2$, it is proved that

$$E_{g,n}(\text{tr} \gamma) = \frac{2}{n} + \frac{5}{n^3} + O(n^{-5}).$$

More asymptotic formulas are proved in this paper in terms of the Witten zeta-function and unitary dual of $\text{SU}(n)$. Every irreducible $\text{SU}(n)$-representation contributes to the expected values. The calculations heavily use all the machinery of representation theory of $\text{SU}(n)$. Part of their technical difficulty is due to the growth of representations of $\text{SU}(n)$ as $n \nearrow +\infty$. The paper also describes closely related questions in mathematical physics, notably expected values of Wilson loops in 2D Yang-Mills theory.