QUATERNIONS AND EUCLIDEAN 3-SPACE

The algebra of quaternions

Let $\mathbb{H}$ denote a four-dimensional vector space with basis denoted $\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}$. Let $\mathbb{H}_0$ be the 3-dimensional vector space based on $\mathbf{i}, \mathbf{j}, \mathbf{k}$, regarded as vectors in $\mathbb{R}^3$. The *bilinear* map

$$\begin{aligned} \mathbb{H} \times \mathbb{H} &\longrightarrow \mathbb{H} \\ (\mathbf{q}_1, \mathbf{q}_2) &\longmapsto \mathbf{q}_1\mathbf{q}_2 := -(\mathbf{q}_1 \cdot \mathbf{q}_2)\mathbf{1} + (\mathbf{q}_1 \times \mathbf{q}_2) \end{aligned}$$

is called *quaternion multiplcation. Quaternion conjugation* is the *linear* map:

$$\begin{aligned} \mathbb{H} &\longrightarrow \mathbb{H} \\ \mathbf{q} := r\mathbf{1} + \mathbf{q}_0 &\longmapsto \bar{\mathbf{q}} := r\mathbf{1} - \mathbf{q}_0 \end{aligned}$$

where $r \in \mathbb{R}$ is the *real part* $\mathsf{Re}(\mathbf{q})$ and $\mathbf{q}_0 \in \mathbb{H}_0$ is the *imaginary part* $\mathsf{Im}(\mathbf{q})$. Then

$$\mathbf{q}\bar{\mathbf{q}} = \|\mathbf{q}\|^2 = r^2 + \|\mathbf{q}_0\|^2 \geq 0$$

and equals zero if and only if $\mathbf{q} = 0$. Thus if $\mathbf{q} \in \mathbb{H}$ is nonzero, then it is multiplicatively invertible, with its inverse defined by:

$$\mathbf{q}^{-1} := \|qq\|^{-2}\bar{\mathbf{q}}$$

just like for complex numbers.

Thus $\mathbb{H}$ is a *division algebra* (or *noncommutative field.*

The quaternions generalize complex numbers, built from the field $\mathbb{R}$ of real numbers by adjoining *one* root $\mathbf{i}$ of the equation $z^2 = 1$. Note that by adjoining one $\sqrt{-1}$, there is *automatically* a *second* one, namely $-\sqrt{-1}$. This is a special case of the *Fundamental Theorem of Algebra,* that (counting mulitiplicities) a polynomial equation of degree n admits n complex roots.

However, there is no ordering on the field $\mathbb{C}$ of complex numbers, that is, there is no meaningful sense of a "positive" or "negative" complex number. Thus there is no essential difference between $\mathbf{i}$ and $-\mathbf{i}$. This *algebraic symmetry* gives rise to the field automorphism of *complex conjugation:*

$$\begin{aligned} \mathbb{C} &\longrightarrow \mathbb{C} \\ z &\longmapsto \bar{z} \end{aligned}$$

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The quaternions arise by adjoining *three* values of $\sqrt{-1}$, each in one of the coordinate directions of $\mathbb{R}^3$. Thus we obtain 6 values of $\sqrt{-1}$, but in fact there are *infinitely many* square-roots of -1, one in *every* direction in $\mathbb{R}^3$.

However, these basic quaternion don't commute, but rather *anti-commute:*

$$\begin{aligned}\mathbf{ij} &= -\mathbf{ji} = \mathbf{k}\\ \mathbf{jk} &= -\mathbf{kj} = \mathbf{i}\\ \mathbf{ki} &= -\mathbf{ik} = \mathbf{j}\end{aligned}$$

Recall that (multi)linear maps of vector spaces can be uniquely determined by their values on a basis. These can be succinctly expressed in terms of *tables* as follows. Multiplication tables for the dot and cross products of vectors in $\mathbb{R}^3 = \mathbb{H}_0$ are:

$\cdot$	**i**	**j**	**k**
i	1	0	0
j	0	1	0
k	0	0	1

$\times$	**i**	**j**	**k**
i	0	**k**	$-\mathbf{j}$
j	$-\mathbf{k}$	0	**i**
k	**j**	**-i**	0

We can describe quaternion operations by their *tables* as they are multilinear. For example, quaternion conjugation is described in the basis $(\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k})$ as:

1	**1**
i	**-i**
j	**-j**
k	**-k**

Here is the multiplication table for quaternion multiplication:

	$\mathbf{1}$	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
$\mathbf{i}$	$\mathbf{i}$	$-\mathbf{1}$	$\mathbf{k}$	$-\mathbf{j}$
$\mathbf{j}$	$\mathbf{j}$	$-\mathbf{k}$	$-\mathbf{1}$	$\mathbf{i}$
$\mathbf{k}$	$\mathbf{k}$	$\mathbf{j}$	$-\mathbf{i}$	$-\mathbf{1}$

A *unit quaternion* is a quaternion $\mathbf{q} \in \mathbb{H}$ such that $\|\mathbf{q}\| = 1$. Unit quaternions form the *unit* 3*-sphere* $S^3 \subset \mathbb{R}^4$. The *imaginary unit quaternions* $\mathbb{H}_1$ form a 2-sphere

$$S^2 \subset \mathbb{H}_0 = \mathbb{R}^3.$$

Note that if $\mathbf{u} \in \mathbb{H}_1$ is an imaginary unit quaternion then $\mathbf{u}^2 = -1$. This gives the *infinitely many* square-roots of -1 promised earlier. Furthermore, since

$$\mathbf{u}^n = \begin{cases} 1 & \text{if } n \equiv 0 (\mathsf{mod} 4) \\ \mathbf{u} & \text{if } n \equiv 1 (\mathsf{mod} 4) \\ -1 & \text{if } n \equiv 2 (\mathsf{mod} 4) \\ -\mathbf{u} & \text{if } n \equiv 3 (\mathsf{mod} 4) \end{cases}$$

the usual calculation with power series implies:

$$\begin{aligned} \exp(\theta\mathbf{u}) &= \sum_{n=0}^{\infty} \frac{1}{n!} (\theta\mathbf{u})^n \\ &= \cos(\theta)\mathbf{1} + \sin(\theta)\mathbf{u} \end{aligned}$$

just like $e^{\mathbf{i}\theta} = \cos(\theta) + \mathbf{i} \sin\theta$ for complex numbers.

Futhermore, if $\mathbf{v} \in \mathbb{H}_0$ represents a vector in $\mathbb{R}^3$, then rotation in the unit vector $\mathbf{u}$ by angle θ is:

$$\mathbf{v} \xrightarrow{\mathsf{Rot}_{\mathbf{u}}^{\theta}} \exp(\theta/2\mathbf{u})\ \mathbf{v}\ \exp(-\theta/2\mathbf{u})$$

The usual Euclidean inner product on $\mathbb{R}^4$ is given in terms of quaternions $\mathbb{H} \cong \mathbb{R}^4$ by:

$$\mathbf{v} \cdot \mathbf{w} = \mathsf{Re}(\mathbf{v}\bar{\mathbf{w}}),$$

again, just like the analogous formula for complex numbers.

Remarks on 3×3 skew-symmetric matrices

The geometry of Euclidean 3-space $\mathbb{R}^3$ closely relates to the cross-product and skew-symmetric 3×3-matrices. Namely, if

$$\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k},$$

then the linear map

$$\mathbf{w} \longmapsto \mathbf{v} \times \mathbf{w}$$

is represented by the skew-symmetric matrix

$$S(\mathbf{v}) = S(x, y, z) := \begin{bmatrix} 0 & z & -y \\ -z & 0 & x \\ y & -x & 0 \end{bmatrix}$$

in the standard basis $(\mathbf{i}, \mathbf{j}, \mathbf{k})$. Clearly every 3×3 skew-symmetric matrix is $S(x, y, z)$ for some $(x, y, z) \in \mathbb{R}^3$.

This establishes an isomorphism

$$\mathfrak{so}(3) \cong \mathbb{R}^3 \cong \mathbb{H}_0$$

where $\mathfrak{so}(n)$ denotes the space of $n \times n$ skew-symmetric matrices.

Notice that $\mathsf{Det} S(\mathbf{v}) = 0$ (this follows because $\mathbf{v} \in \mathsf{Ker} S(\mathbf{v})$ so that $S(\mathbf{v})$ is singular). Computing $S(x_1, y_1, z_1) S(x_2, y_2, z_2)$, establishes the useful formula

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{c} \cdot \mathbf{a})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$$

which implies that

$$S(qv)^2 \;=\; -\|\mathbf{v}\|^2 \Pi_{\mathbf{v}^\perp} \;\oplus\; 0 \Pi_{\mathbf{v}}$$

where Π denotes orthogonal projection. In particuar the eigenvalues of $S(\mathbf{v})$ are $0, \pm\|\mathbf{v}\|$.

Furthermore

$$S(\mathbf{v})S(\mathbf{w}) - S(\mathbf{w})S(\mathbf{v}) = 2S(\mathbf{v} \times \mathbf{w})$$

Another consequence of this formula is *Jacobi's identity:*

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = 0$$