Dr. Wolfe MATH 240 MATLAB PROJECT #1 Due October 14, 2004

1. Read section 2.5 of *Lay* at least through the material in the box on p.146. Then do problem 31 on p.150. In part (a) use the command **[L,U]=lu(A)**. In part (b) use the backslash operator twice. (See equation (2) on p.134.)

MATLAB uses finite precision arithmetic, so there will be round-off error in most calculations. In order to minimize the effect of round-off error the Gaussian elimination process must be modified by introducing the concept of *partial pivoting*. (The definition is given on p.20 of *Lay*.) The MATLAB command $A\backslash\mathbf{b}$ impliments Gaussian elimination with partial pivoting. What accuracy should one expect ?

MATLAB carries about 15 digits so its basic calculations are very accurate, with errors about 10^{-15}. We can quantify this more precisely. The numbers that can be represented in MATLAB are called floating point numbers. There is a special number in MATLAB denoted by *eps* , that is defined to be the distance from 1 to the next floating point number. The command **eps** will produce *eps* for the computer you are using; *eps* is approximately $2.22e - 16$. The significance of *eps* is that it determines the round-off error committed when a number is rounded: If a positive number a is rounded, the round-off error is approximately $a \times eps$.

Consider the linear system

$$A\mathbf{x} = \mathbf{b},$$

and let $\mathbf{x}^*$ be the computed solution (the number produced by the computer). There are two measures of accuracy of the computed solution $\mathbf{x}^*$. The *error* in $\mathbf{x}^*$ is defined to be

$$\mathbf{e} = \mathbf{x} - \mathbf{x}^*,$$

and the *residual* is defined to be

$$\mathbf{r} = \mathbf{b} - A\mathbf{x}^*.$$

The *norm* (size) of $\mathbf{e}$ is one measure of the accuracy of $\mathbf{x}^*$; if the norm of $\mathbf{e}$ is small, then $\mathbf{x}^*$ is close to $\mathbf{x}$. The second measure of accuracy of $\mathbf{x}^*$ is the norm of $\mathbf{r}$; if the norm of $\mathbf{r}$ is small, then $\mathbf{x}^*$ nearly satisfies the linear system.

There are two principles that indicate the accuracy of the solution $\mathbf{x}^*$ produced by Gaussian elimination with partial pivoting (the method used in $A\backslash\mathbf{b}$). The first is that the norm of the residual is nearly always small; to be more precise,

$$\text{Norm of } \mathbf{r} \propto \text{Norm of } \mathbf{x} \times \text{Norm of } A \times eps,$$

i.e., for a moderate sized problem the norm of the residual is approximately *eps* . The second principle is that

$$\text{Norm of } \mathbf{e} \propto \text{Norm of } \mathbf{x} \times \text{cond}(A) \times eps,$$

where Cond(A) is a number that measures how close A is to being singular. The norm of a vector or a matrix is a measure of the size of the vector or the matrix, respectively. If A

is nearly singular, then Cond(A) is large, and the error may be large. Matrices for which cond(A) is large are called *ill-conditioned.* If Cond(A) $\approx 10^t$, then the error would be about $10^t \times eps \approx 10^{16-t}$, *i.e.*, we would have lost approximately t digits of accuracy. Cond(A) can be found with the command **cond(A)**. When $A\backslash\mathbf{b}$ is used, if the calculations indicate that A is ill-conditioned, a warning message is printed, and a number RCOND which is an approximation to $1/\text{cond}(A)$, is printed. The following two problems illustrate these principles. Note: type **format long** before you do the requested calculations.

2. The $n \times n$ matrix H_n whose elements are given by $h_{i,j} = 1/(i+j-1)$ is very ill-conditioned if n is large; it is called the *Hilbert Matrix.* It can be generated in MATLAB with the command **hilb(n)**. Let A be the Hilbert matrix of size 11. Let $\mathbf{x}$ =**ones(11,1)**, and let $\mathbf{b} = A\mathbf{x}$. Now solve the system $A\mathbf{x} = \mathbf{b}$, obtaining $\mathbf{x}^*$. Since we know $\mathbf{x}$ exactly we can compute $\mathbf{e}$. Do this and also compute $\mathbf{r}$. What are the norms of these vectors ? Are the above principles satisfied for this example ?

3. Let
$$A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 6 & 3 & -2 \\ 7 & 8 & 9 & -3 & 3 \\ 3 & 4 & -2 & -4 & 10 \\ 1 & 2 & 3 & 4 & 6 \end{pmatrix}.$$
Find cond(A). A is not ill-conditioned. Solve $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b} = (55, 34, 53, 39, 60)^T$. How small would you expect $\mathbf{e}$ and $\mathbf{r}$ to be ? The exact solution is $\mathbf{x} = (1, 2, 3, 4, 5)^T$. Calculate $\mathbf{e}$ and $\mathbf{r}$. Are the above principles satisfied for this example ?