MATH 240 **FINAL EXAM** **May 16, 2003**

Instructions: Number the answer sheets from 1 to 9. Fill out all the information at the top of each sheet. Answer problem n on page n, $n = 1, \cdots, 9$. Do not answer one question on more than one sheet. If you need more space use the back of the correct sheet. Please write out and sign the **Honor Pledge** on page 1 only.

SHOW ALL WORK

1. (30 points) Let

$$A = \begin{pmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{pmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 7 \\ -1 \\ -4 \end{bmatrix}.$$

(a) Is $\mathbf{b}$ in col(A) (the column space of A) ?
(b) Is $\mathbf{b}$ in nul(A) (the nullspace of A) ?
(c) Find a basis for col(A).
(d) Find a basis for nul(A).
(e) What is the rank of A ?
Note: Please work carefully. If you answer a question incorrectly due to arithmetic errors you will lose most or all of the credit on that part.

2. (25 points) Let

$$B = \begin{pmatrix} 3 & 1 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Denote the columns of B by $\mathbf{b_1}, \mathbf{b_2}, \mathbf{b_3}$.
(a) Show that $\{\mathbf{b_1}, \mathbf{b_2}, \mathbf{b_3}\}$ is a basis for $\mathbf{R^3}$.
(b) Let $\mathbf{u} = (2, 1, 3)^T$. Determine the coordinate vector of $\mathbf{u}$ relative to the basis $\{\mathbf{b_1}, \mathbf{b_2}, \mathbf{b_3}\}$ of $\mathbf{R^3}$.
(c) Determine if B is diagonalizable. (Mention one or more theorems.)

3. (20 points)
(a) Complete the following definition: A subset S of a vector space V is a *Subspace of* V if

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(b) Let V be the subset of $\mathbf{R^2}$: $V = \{(x_1, x_2)^T |\ x_1 x_2 = 0\}$. Prove or disprove the statement "V is a subspace of $\mathbf{R^2}$".

4. (20 points) By reviewing its donation records, the alumni office of a college finds that 80% of its alumni who contribute to the annual fund one year will also contribute the next year, and 30% of those who do not contribute one year will contribute the next.
(a) Write down P, the stochastic matrix for this situation.
(b) Over the long term, what percentage of the alumni will contribute ?
(c) What is $\lim_{n\to\infty} P^n$?

5. (19 points)
   (a) Suppose (x_1, x_2, x_3) is the solution of the system

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 4 \\ 3x_1 + 2x_2 + 4x_3 &= 7 \\ 4x_1 + 2x_2 + 3x_3 &= 7 \end{aligned}$$

   Express x_2 as a quotient (ratio) of determinants. Do not evaluate the determinants.
   (b) Find the area of the parallelogram with vertices at $(-1, 0)$, $(0, 5)$, $(1, -4)$, $(2, 1)$.
   (c) Suppose that A is a square matrix such that $\det(A^4) = 0$. Explain why A cannot be invertible.

6. (20 points) Let

$$A = \begin{pmatrix} 5 & -5 \\ 1 & 1 \end{pmatrix}$$

   Let A act on $\mathbf{C^2}$. Find the eigenvalues and a basis for each eigenspace in $\mathbf{C^2}$.

7. (15 points) Find the distance from the point $(4, -7)$ to the line $x_2 = 2x_1$. Draw a picture to show what you are doing.

8. (30 points) Let

$$A = \begin{pmatrix} 13 & 4 & -2 \\ 4 & 13 & -2 \\ -2 & -2 & 10 \end{pmatrix}.$$

   The characteristic polynomial of A is $p(\lambda) = (18 - \lambda)(\lambda - 9)^2$.
   (a) Find an orthogonal matrix P such that $P^T AP$ is diagonal.
   (b) Classify the quadratic form $Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$ as positive definite, negative definite, or indefinite.
   (c) Find the maximum of $Q(\mathbf{x})$ subject to the constraint $\|\mathbf{x}\| = 1$.

9. (21 points) Mark each statement as true (T) or false (F).
   (a) If S is a linearly dependent set of vectors, then each vector is a linear combination of the other vectors in S.
   (b) If A and B are $n \times n$ matrices, then $(A + B)(A - B) = A^2 - B^2$.
   (c) Let A be an $n \times n$ matrix. If B is formed by adding to one row of A a linear combination of the other rows, then $\det B = \det A$.
   (d) If B is obtained from a matrix A by several elementary row operations, then $\text{rank}\, A = \text{rank}\, B$.
   (e) If A is diagonalizable, then the columns of A are linearly independent.
   (f) If a square matrix has orthonormal columns, then it also has orthonormal rows.
   (g) A positive definite quadratic form Q satisfies $Q(\mathbf{x}) > 0$ for all $\mathbf{x}$ in $\mathbf{R^n}$.