MATH 401 Dr. Wolfe ASSIGNMENT #4 Due December 8, 2003

1. Use the program `pix.m` which you can find on the class webpage. You have three data choices. You must also enter the number of subdivisions in the mesh. Then you can approximate the original image by adding up to N terms of the singular value expansion of the image matrix.

   (a) Enter 10 for the mesh size. Then choose data choice 1. You will see a random 10×10 pattern of gray, white and black squares. This is the original picture. Now you can approximate this picture by adding up the terms of the SVD expansion of the matrix. How many terms in the expansion are necessary to get a recognizable approximation? Note how fast the singular values are decreasing.

   (b) Enter 40 for the mesh size. Then choose data choice 2. Now how many terms are necessary to get a good approximation of the original image? Compare the rate of decrease of the singular values of this image with the rate of decrease in part a). If you use only 4 terms in the expansion, how much do you save in storage space as compared to the storge space for the full 40×40 matrix?

   (c) Enter 100 for the mesh size. Choose data choice 3. This is as close as I can come to a fingerprint. Again note how many terms are necessary to get a good approximation. Relate this observation to the rate at which the singular values decrease. Compare with the rates of parts a) and b). How much storage space do you save if you use 20 terms in the expansion as compared to the full 100×100 matrix ?

   (d) Make up your own image matrix Z and insert it in the line for data choice 4. Analyse it as in question 1.

2. The eigenvalue problem

$$-\frac{d^2u}{dx^2} = \lambda u,\ 0 < x < 1, \quad u(0) = u(1) = 0$$

has the eigenvalues $\lambda_n = n^2\pi^2$ and the corresponding eigenfunctions $u_n(x) = \sin n\pi x$. If we discretize the problem as we did with the boundary value problem in Assignment #1 we get

$$-u_{j-1} + 2u_j - u_{j+1} = h^2\lambda u_j,\ j = 1, \ldots, n \tag{1}$$

where $u_j \approx u(jh)$ and $h = \frac{1}{n+1}$.

Use MATLAB to find the eigenvalues of the matrix in (1) for $n = 9, 19, 29, 39$. In each case find the smallest eigenvalue $\tilde{\lambda}_1$ and compare $\tilde{\lambda}_1/h^2$ with $\lambda_1 = \pi^2$. Also take the corresponding eigenvector, rescale it so that the largest entry is 1 and compare it graphically with $u_1 = \sin \pi x$.

3. (MATLAB) Let

$$A = \begin{bmatrix} 7 & -4 & 4 \\ -4 & 5 & 0 \\ 4 & 0 & 9 \end{bmatrix}.$$

(a) Use the power method to find the dominant eigenvalue of A.
(b) Use the unshifted QR algorithm: $A_0 = A, A_k = Q_k R_k, A_{k+1} = R_k Q_k$ to find the eigenvalues of A.
(c) Use the explicitly shifted QR algorithm: $A_0 = A, A_k - A_k(3,3)I = Q_k R_k, A_{k+1} = R_k Q_k + A_k(3,3)I$ to find the eigenvalues of A. When the third row and column of A_k are essentially zero except for the (3,3) entry 'deflate' the problem by working with the 2×2 principal submatrix.

4. Consider the system

$$\begin{aligned} 4x_1 + x_2 + x_3 + x_4 &= -5 \\ x_1 + 8x_2 + 2x_3 + 3x_4 &= 23 \\ x_1 + 2x_2 - 5x_3 \qquad &= 9 \\ -x_1 \qquad + 2x_3 + 4x_4 &= 4 \end{aligned}$$

Use both the Jacobi method and the Gauss-Seidel method to solve the system. Take $\mathbf{x}^{(0)} = \mathbf{0}$, and terminate iteration when $\|\mathbf{x}^{k+1} - \mathbf{x}^k\|$ falls below 5×10^{-7}. Record the number of iterations required to achieve convergence.

5. Maximize $z = 3x_1 - 2x_2 - x_3 + x_4$ subject to

$$\begin{aligned} 2x_1 - 3x_2 + x_3 - x_4 &\leq 6 \\ x_1 + 2x_2 - x_3 + 2x_4 &\leq 4 \end{aligned}$$

and

$$x_1, x_2, x_3, x_4 \geq 0.$$

(answer: $x_1 = 16/5, x_2 = 0, x_3 = 0, x_4 = 2/5; z = 10.$)