MATH 401 SAMPLE EXAM I

1. Let
$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 1 \\ 3 & 2 & 1 \end{pmatrix}$$
   (a) Find a permutation matrix P, a lower triangular matrix L and an upper triangular matrix U such that
$$PA = LU. \tag{1}$$
   (b) Show how the factorization (1) is used to solve $A\mathbf{x} = \mathbf{b}$.

2. Let $\mathbf{u} = (1,1,1,1)^T,\ \mathbf{v} = (1,7,1,7)^T,\ W = \text{Span}\{\mathbf{u},\mathbf{v}\}$.
   (a) Calculate $\|\mathbf{v}\|$, the projection of $\mathbf{v}$ onto $\mathbf{u}$ and the unit vector in the direction of $\mathbf{u}$.
   (b) Apply the Gram-Schmidt process to $\{\mathbf{u},\mathbf{v}\}$ to obtain an orthonormal basis for W.
   (c) Let $\mathbf{y} = (3,2,-1,2)^T$. Find $\mathbf{z}$, the vector in W which is closest to $\mathbf{y}$.

3. Let $\mathbf{u_1} = (1,2,3)^T$, $\mathbf{u_2} = (1,1,-1)^T$, $W = \text{span}\{\mathbf{u_1},\mathbf{u_2}\}$. Find a basis for $W^{\perp}$.

4. Let $\mathbf{u}$ be a unit vector in $\mathbf{R}^n$ and let $P = I - 2\mathbf{u}\mathbf{u}^\mathbf{T}$. ($P$ is an $n \times n$ matrix.) Show
   (a) P is symmetric.
   (b) P is orthogonal.
   (c) $P^2 = I$.
   If $\mathbf{u} = (\frac{3}{5},\frac{4}{5})^T$, what is P ?

5. We wish to fit the data (0,1), (1,3), (2,7), (3,10), (4,20) to a function of the form
$$f(x) = a + bx + ce^x$$
   in the sense of least squares. Find an equation for the coefficients a, b and c. Do not do any computations.

6. In $C[0,1]$ with the inner product defined by
$$f \cdot g = \int_0^1 f(x)g(x)\,dx$$
   consider the vectors 1 and x.
   (a) Determine the projection p of 1 onto x and verify that $1-p$ is orthogonal to p.
   (b) Compute $\|1-p\|, \|p\|, \|1\|$ and verify that the Pythagorean theorem holds.