MATH 401 SAMPLE FINAL EXAM

1. (15 points) Let

$$A = \begin{pmatrix} 36 & 30 & 18 \\ 30 & 41 & 23 \\ 18 & 23 & 12 \end{pmatrix}$$

   (a) Factor A as $A = LU$ where L is unit lower triangular (i.e. $l_{ii} = 1, i = 1, 2, 3.$) and U is upper triangular. (This is the usual LU decomposition).
   (b) Now factor A as $A = LDL^T$ where L is as in part (a) and D is diagonal.
   (c) Use the result of (b) to show that A is not positive definite. Can you find a vector $\mathbf{x} \in R^3$ such that $\mathbf{x^T} A\mathbf{x} < 0$?

2. (18 points) Let

$$C = \begin{pmatrix} 2 & 4 \\ 2 & 10 \\ 1 & -1 \end{pmatrix}$$

   (a) Apply the Gram-Schmidt procedure to find an orthonormal basis for the column space of C.
   (b) Use the result of (a) to find a 3×2 matrix Q with orthonormal columns and a 2×2 upper triangular matrix R such that $C = QR$.
   (c) Use the result of (b) to find the least squares solution to $C\mathbf{x} = \mathbf{b}$ where $\mathbf{b} = (18, 12, 12)^T$.

3. (15 points) Let

$$B = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

   (a) Show that B is not diagonalizable.
   (b) Compute e^{tB}.

4. (12 points) Find the general solution in terms of real functions of the system

$$y_1' = y_1 + y_2$$
$$y_2' = -2y_1 + 3y_2$$

5. (12 points) Let

$$A = \begin{pmatrix} 5 & 2 \\ -2 & 0 \end{pmatrix}$$

   (a) Use the power method to find an approximation to the dominant eigenvalue of A. To be more specific, let $\mathbf{q_0} = (1, 1)^T$ and compute $\mathbf{q_1}, \mathbf{q_2}, \mathbf{q_3}, \mathbf{q_4}$, $\mathbf{q_k} = A^k\mathbf{q_0}$. It should be clear that if we rescale the **q's** appropriately, the resultant vectors converge to a vector $\mathbf{x_1}$. What is $\mathbf{x_1}$? Show that $\mathbf{x_1}$ is an eigenvector of A and find the corresponding eigenvalue λ_1.

(b) Find an orthogonal matrix U whose first column is an eigenvector of A corresponding to λ_1. You should be able to find U by inspection. Compute $B = U^T AU$. This should be upper triangular. Explain how you can now find the other eigenvalue of A by inspection.

6. (18 points) A carpenter can make tables and chairs. Each table takes 2 hours to make and uses 30 board-feet of lumber. A chair takes one hour to make and uses 10 board-feet of lumber. Each table brings a profit of \$25 and each chair brings a profit of \$10. He has 90 board-feet of lumber on hand. He wants to work no more than 8 hours and maximize his profit.
   (a) Formulate this situation as a Linear Programming problem.
   (b) Solve the problem by graphing the feasible set for the problem.
   (c) Solve the problem by using the simplex algorithm.

7. (10 points) Mark each of the following statements true (T) or false (F). If the statement is false explain why it is false.
   (a) Every matrix A has a singular value decomposition.
   (b) If $\mathbf{u}$ and $\mathbf{v}$ are vectors in R^n, the orthogonal projection $\mathbf{p}$ of $\mathbf{u}$ on $\mathbf{v}$ is $\mathbf{p} = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{v}$.
   (c) Let A be an $n \times n$ matrix. If A is diagonalizable then A is invertible.
   (d) Let M be a Markov matrix (sometimes called a stochastic matrix). Then 1 is an eigenvalue of M.
   (e) The graph of $x^2 + 4xy + 5y^2 = 1$ is a hyperbola.