MATH 410 Dr. Wolfe SAMPLE EXAM I Fall 2006

In this exam, all sets are subsets of $\mathbf{R}$. $N$ is the set of natural numbers (positive integers). You may use any result proven in class.

1. (15 points) Use induction to show that for all all $n \in N$,

$$\sum_{k=1}^{n}(2k-1) = n^2.$$

2. (20 points) Recall that we say that the infinite series $\sum_{k=1}^{\infty} a_k$ is convergent if the sequence of partial sums $\{s_n\}$ where $s_n = \sum_{k=1}^{n} a_k$ converges. Suppose that

$$0 \le a_k \le b_k \text{ for all } k \in N.$$

Show that if $\sum_{k=1}^{\infty} b_k$ is convergent so is $\sum_{k=1}^{\infty} a_k$. (This is the comparison test.)

3. (20 points) Let $f : [a, b] \to \mathbf{R}$ be continuous. Suppose also that

$$a \le f(x) \le b \text{ for all } x \in [a, b].$$

(We say that $f([a, b]) \subset [a, b]$.) Prove that there is a $c \in [a, b]$ such that

$$f(c) = c.$$

Hint: Draw a picture.

4. (25 points) A function $f : D \to \mathbf{R}$ is said to be <u>Lipschitz continuous</u> if for some $K > 0$

$$|f(x) - f(y)| \le K|x - y| \text{ for all } x, y \in D.$$

   (a) Show that if $f$ is Lipschitz continuous on $D$, it is uniformly continuous on $D$.
   (b) Define $f(x) = \sqrt{x}$ for $0 \le x \le 1$. Prove that the function $f : [0, 1] \to \mathbf{R}$ is uniformly continuous but not Lipschitz continuous.

5. (20 points) A set $F$ is <u>closed</u> if it contains its limit points. A set $\mathcal{O}$ is <u>open</u> if for every $x \in \mathcal{O}$ there is a $\delta_x > 0$ such that $(x - \delta_x, x + \delta_x) \subset \mathcal{O}$.
   (a) Show that $[0, 1]$ is closed, $(0, 1)$ is open and $(0, 1]$ is neither open nor closed.
   (b) Is there a non-empty set which is <u>both</u> open and closed?