MATH 411 Dr. Wolfe ASSIGNMENT #1 Due September 20, 2004

1. Ex.8, p.260, *Fitzpatrick.*
2. Given numbers a, b, c with $c > 0$:
   (a) Show that $ab \leq \frac{1}{2}(a^2/c + cb^2)$. Hint: consider $(a/c^{1/2} - c^{1/2}b)^2$.
   (b) Show that $|ab|^{1/2} \leq \frac{1}{2}(|a| + |b|)$.
3. For $\mathbf{p} = (x_1, \ldots, x_n) \in \mathbf{R^n}$ consider the new "taxi-cab" norm defined by

$$\|\mathbf{p}\|_1 = |x_1| + \ldots + |x_n|$$

   (a) Show that $\|\mathbf{p}\| \leq \|\mathbf{p}\|_1 \leq n^{1/2}\|\mathbf{p}\|$. Hint: consider squares and use 2(a).
   (b) Given $(\mathbf{p_k}) \subset \mathbf{R^n}$ show that $\mathbf{p_k} \to \mathbf{p}^* \Leftrightarrow \|\mathbf{p_k} - \mathbf{p}^*\|_1 \to 0$.
   (c) Sketch $S_1 = \{\mathbf{p} \in \mathbf{R^2} : \|\mathbf{p}\|_1 < 1\}$.
4. For $\mathbf{p} = (x_1, \ldots, x_n) \in \mathbf{R^n}$ consider the uniform norm defined by

$$\|\mathbf{p}\|_\infty = \max\{|x_1|, \ldots, |x_n|\}.$$

   (a) Show that $\|\mathbf{p}\|_\infty \leq \|\mathbf{p}\| \leq n^{1/2}\|\mathbf{p}\|_\infty$.
   (b) Show that $\|\mathbf{p}\|_\infty \leq \|\mathbf{p}\|_1 \leq n\|\mathbf{p}\|_\infty$ (See Ex. 3.)
   (c) Given $(\mathbf{p_k}) \subset \mathbf{R^n}$ show that $\mathbf{p_k} \to \mathbf{p}^* \Leftrightarrow \|\mathbf{p_k} - \mathbf{p}^*\|_\infty \to 0$.
   (d) Sketch $S_\infty = \{\mathbf{p} \in \mathbf{R^2} : \|\mathbf{p}\|_\infty < 1\}$.
5. Given $S = \{(1/k, 1/m) \in \mathbf{R^2} : k, m > 0, \text{integers}\}$ determine ∂S, the boundary of S.
6. Starting at the origin in the plane move 1 unit north, then 1/2 unit east, then 1/4 unit south, then 1/8 unit west, then 1/16 unit north, then 1/32 unit east and so on. Determine the point you are approaching.
7. Provide a counterexample to the following statement: If $(\mathbf{p_k})$ is bounded and $(\|\mathbf{p_k}\|)$ is increasing then $(\mathbf{p_k})$ converges.
8. We can generalize the Nested Interval Thoerom in the following manner. Let $\{K_j\}$ represent a collection of non-empty compact sets where $K_{j+1} \subset K_j$ for all $j > 1$. Show that

$$\cap_1^\infty K_m \neq \emptyset.$$

   (Select $(\mathbf{p_j})$ with $\mathbf{p_j} \in K_j$ for all $j \geq 1$.)