MATH 411 Dr. Wolfe ASSIGNMENT #3 Due October 18, 2004

1. Consider $f : \mathbf{R^2} \to \mathbf{R}$ given by $f(x, y) = 0$ whenever $y \leq 0$ or $y \geq x^2$ and $f(x, y) = 1$ for $0 < y < x^2$. For any fixed value of α show that $f(x, \alpha x) \to 0$ as $x \to 0$ (f has limit zero along any line). However, determine a sequence $(x_k, y_k) \to (0, 0)$ where $f(x_k, y_k) \to 1$ (so that f is discontinuous at the origin).
2. Consider $f : \mathbf{R^n} \to \mathbf{R}$ such that $f(\mathbf{p_1} + \mathbf{p_2}) = f(\mathbf{p_1})f(\mathbf{p_2})$ for all $\mathbf{p_1}, \mathbf{p_2} \in \mathbf{R^n}$.
   (a) Show that either $f(\mathbf{0}) = 0$ or $f(\mathbf{0}) = 1$.
   (b) Give a (non-constant) example of such an f with $f(0) = 1$ when $n = 1$.
   (c) Assume $f(\mathbf{0}) = 1$. Show that f is continuous on $\mathbf{R^n}$ if and only if f is continuous at $\mathbf{0}$.
3. Consider $f(x, y) = |x - y|$. Determine all directions $\mathbf{p}$ for which $\frac{\partial f}{\partial \mathbf{p}}(1, 1)$ exists.
4. Consider f defined by

$$f(x_1, \ldots, x_n) = \begin{cases} x_1 x_2 \cdots x_n / (x_1^2 + x_2^2 + \cdots x_n^2), & (x_1, \ldots, x_n) \neq (0, \ldots, 0) \\ 0, & (x_1, \ldots, x_n) = (0, \ldots, 0) \end{cases}$$

   Determine all values of n for which f is differentiable.
5. (Rolle's Theorem) Suppose $f : \mathbf{R^n} \to \mathbf{R}$ is a differentiable function such that $f(\mathbf{p}) = 0$ if $\|\mathbf{p}\| = 1$. Show that there is a $\mathbf{q}$ with $\|\mathbf{q}\| < 1$ such that $\mathbf{D}f(\mathbf{q}) = \mathbf{0}$. (For $n = 1$, this is Rolle's theorem. Examine the proof in this case.)
6. Ex.7, p.335, *Fitzpatrick.*
7. Ex.18, p.336, *Fitzpatrick.*
8. Ex.3, p.350, *Fitzpatrick.*