MATH 411 Dr. Wolfe ASSIGNMENT #4 Due November 3, 2004

1. Let $\mathbf{p_1} = (3/5, -4/5), \mathbf{p_2} = (3/5, 4/5)$. If f is differentiable at $(1,1)$ and $\frac{df}{d\mathbf{p_1}}(1,1) = 3$, $\frac{df}{d\mathbf{p_2}}(1,1) = 2$, find the maximum value for $\frac{df}{d\mathbf{p}}(1,1)$ for $\|\mathbf{p}\| = 1$.

2. Assume $f : \mathbf{R^2} \to \mathbf{R}$ is differentiable on $\mathbf{R^2}$.
   (a) If $f(-1,0) = 0$ and $f(1,0) = 1$ show that there exist points $\mathbf{p_1}, \mathbf{p_2}$ with $\mathbf{p_1} \neq \mathbf{p_2}$, $\|\mathbf{p_1}\| = \|\mathbf{p_2}\| = 1$ and $f(\mathbf{p_1}) = f(\mathbf{p_2}) = 1/2$.
   (b) If $f(0,0) = 0$ and $f(\mathbf{p}) \geq 0$ for all $\|\mathbf{p}\| = 1$, show there exists a $\mathbf{q}$ such that $\|\mathbf{q}\| < 1$ and $\mathbf{D}f(\mathbf{q}) = \mathbf{0}$.

3. Assume $f : \mathbf{R^2} \to \mathbf{R}$ is differentiable on $\mathbf{R^2}$.
   (a) Assume $f(0,0) = 0$ and define $g : \mathbf{R^2} \to \mathbf{R}$ by $g(x,y) = f(f(y,x), f(-x,y))$. Show that $g_x(0,0) = 0$ and $g_y(0,0) = \|\mathbf{D}f(0,0)\|^2$. Also, verify these results directly when $f(x,y) = \alpha x + \beta y$; α, β constant.
   (b) Define $u(x,y) = f(x/y, y/x)$ with domain $\mathcal{D} = \{(x,y) : xy \neq 0\}$. Show that $xu_x + yu_y = 0$ for all $(x,y) \in \mathcal{D}$.
   (c) Define $v : \mathbf{R^2} \to \mathbf{R}$ by $v(x,y) = f(x-y, y-x)$. Show that $v_x + v_y = 0$.
   (d) Define $p : \mathbf{R} \to \mathbf{R}$ by $p(t) = f(f(t,-t), f(t,t))$ and assume $f(0,0) = 0$. Show that $p'(0) = \|\mathbf{D}f(0,0)\|^2$ and verify this directly when $f(x,y) = \alpha x + \beta y$; α, β constant.

4. Assume $u : \mathbf{R^2} \to \mathbf{R}$ is twice continuously differentiable on $\mathbf{R^2}$
   (a) If u satisfies the partial differential equation

$$x^2 u_{xx} + y^2 u_{yy} + xu_x + yu_y = 0 \tag{1}$$

   show that the change of variables $x = e^s$, $y = e^t$ transforms (1) into

$$u_{ss} + u_{tt} = 0$$

   (b) Show that u satisfies the PDE $au_x + bu_y = 0$ where $ab \neq 0$ if and only if u is of the form $u(x,y) = g(ay - bx)$ where g is smooth. Hint: Consider the change of variables $x = as - bt, y = bs + at$).

5. Assume $u : \mathbf{R^2} \to \mathbf{R}$ is smooth on $\mathbf{R^2}$. By employing polar co-ordinates show that :
   (a) $xu_x + yu_y = 0$; $(x,y) \neq (0,0)$ if and only if $u(x,y) = F(\theta)$ for a smooth function F.
   (b) $yu_x - xu_y = 0$; $(x,y) \neq (0,0)$ if and only if $u(x,y) = G(r)$ for a smooth function G.

6. Consider $f(x,y) = x^2 y + x + y$. Find θ for which the Mean Value Theorem (Theorem 13.9 in the book) applied to f is satisfied when $\mathbf{x} = (0,0)$, $\mathbf{h} = (1,1)$.

7. Ex.11, p.379, *Fitzpatrick.*