MATH 411 Dr. Wolfe ASSIGNMENT #5 Due November 15, 2004

1. The curve $x^3 + y^3 = 3xy$ is the *folium of Descartes*. Find the equation of the tangent line to the curve at $(3/2, 3/2)$.

2. Consider the sphere, S, centered at (x^*, y^*, z^*) with radius $r > 0$. We wish to determine all points $(x_0, y_0, z_0) \in S$ for which all points on S in a neighborhood of (x_0, y_0, z_0) can be expressed in the form $z = g(x, y)$.
   (a) Show geometrically there exists a circle C for which this is impossible.
   (b) Verify (a) algebraically.
   (c) Find C by using the Implicit Function Theorem.

3. The intersection of the plane $x + y + z = 1$ and the sphere $x^2 + y^2 + z^2 = 3$ forms a circle C. Verify that the points on C in a neighborhood of $(-1/3, -1/3, 5/3)$ can be parameterized by one of the variables.

4. The plane $3x + 4y + z = 4$ intersects the cone $z = \sqrt{x^2 + y^2}$ in a hyperbolic curve H. Note that $(1, 0, 1) \in H$. If $(x, y, z) \in H$ is sufficiently close to $(1, 0, 1)$ and $z > 1$, show that $x > 1$ and $y < 0$.

5. Assume $g : \mathbf{R} \to \mathbf{R}$ is smooth and consider the equation:

$$g(xy) = g(x + y) + x - y. \tag{1}$$

   If $|g'(4)| < 1$ show there exists $\epsilon > 0$ such that for all $2 < y^* < 2 + \epsilon$ there exists a unique $x^* > 2$ such that (x^*, y^*) solves (1).

6. Assume $g : \mathbf{R} \to \mathbf{R}$ is smooth, $g(0) = 0$ and consider the equation:

$$z = g(ax + by + cz);\ \text{for fixed, nonzero } a, b, c. \tag{2}$$

   (a) What further condition on g guarantees that we can express (2) in the form $z = \eta(x, y)$ in a neighborhood of $(0, 0, 0)$?
   (b) Assuming (a) show that $b\eta_x(x, y) = a\eta_y(x, y)$ for all (x, y) near $(0, 0)$.

7. A mountain can be expressed by the equation

$$x^2z^2 + 2yz + x^3 + (z^2 + 1)/(x^2 + 1) - x^2y + z^4 + y = 6. \tag{3}$$

   Assume you are standing at $(1, 1, 1)$ and wish to climb down. What horizontal direction should you face so that the instantaneous rate of change of your first step will maximize your descent?

8. For all $(x, y) \approx (1, 1)$ show there exists a unique pair $(u, v) \approx (0, 1)$ where

$$x^2 + yu = vx, \qquad y^2 - xv = u^2.$$

   If $(x, y) \approx (1, 1)$ and $x = y > 1$ is $u > 0$ or $u < 0$?