MATH 411 Dr. Wolfe SAMPLE SECOND HOUR EXAM Fall 2010

1. Suppose $F : \mathbf{R^n} \to \mathbf{R^n}$ is smooth and for some $M > 0$

$$\|F(\mathbf{p_1}) - F(\mathbf{p_2})\| \geq M\|\mathbf{p_1} - \mathbf{p_2}\| \text{ for all } \mathbf{p_1}, \mathbf{p_2} \in \mathbf{R^n}.$$

(a) Show that F is 1-1 and F^{-1} is uniformly continuous on $\mathbf{R^n}$.
(b) Show that $J(\mathbf{p}) = \det(DF(\mathbf{p})) \neq 0$ for all $\mathbf{p} \in \mathbf{R^n}$. (Argue by contradiction. Note that if $J(\mathbf{p_0}) = 0$ then there is a $\mathbf{p}^* \neq \mathbf{0}$ such that $DF(\mathbf{p_0})\mathbf{p}^* = \mathbf{0}$.)

2. Let $f : \mathbf{R^3} \to \mathbf{R}$ be a smooth function. Let $w = f(x - y, y - z, z - x)$. Show that

$$\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} + \frac{\partial w}{\partial z} = 0$$

3. Given

$$F(x, y) = (e^{x+y}, e^{x-y}) \text{ for } (x, y) \in \mathbf{R^2}.$$

Apply the Inverse Function Theorem at the point $(x_0, y_0) = (0, 0)$ and calculate the partial derivatives of the components of the inverse mapping at the point $(u_0, v_0) = F(0, 0)$.

4. Consider the system of equations

$$x + 2y^2 + x^2 + (yz)^2 + t^3 = 0$$
$$-x + z + \sin(y^2 + z^2 + t^2) = 0$$

Note that $(x, y, z, t) = (0, 0, 0, 0)$ is a solution. Apply the Implicit Function Theorem to show that in a neighborhood of $(0, 0, 0, 0)$ the equations can (in principle) be solved for two of the variables in terms of the other two. Which two variables can they be solved for? Explain.

5. Use Lagrange multipliers for find the points on the sphere $x^2 + y^2 + z^2 = 1$ at which the function $f(x, y, z) = xyz$ is maximized and mimimized.