AMSC/CMSC 466 SAMPLE FINAL EXAM

1. Recall that in any computation the <u>relative error</u> is defined as

$$\text{rel. error} \;=\; \frac{|\text{ error }|}{|\text{ true value}|}.$$

   We are concerned with the evaluation of a function $f(x)$.
   (a) Show that if the error in x is small

$$\text{rel. error in } f(x) \approx \frac{|xf'(x)|}{|f(x)|} \cdot \text{rel. error in } x.$$

   The quantity $\kappa = |xf'(x)/f(x)|$ can be considered the <u>condition number</u> of f at x. If κ is large we say f is <u>ill conditioned</u> at x. If κ is small we say that f is <u>well conditioned</u> at x.
   (b) Show that for any $x > 0, f(x) = x^{1/2}$ is well conditioned while for x near $\pi/2, f(x) = \cos x$ is ill conditioned. What about $f(x) = \sin x$ near $x = 0$?

2. Let

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 4 \\ 3 & 6 & 7 \end{pmatrix}, \quad \mathbf{b} = \begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix}$$

   (a) Use Gauss elimination to show $A\mathbf{x} = \mathbf{b}$ has no solutions.
   (b) Prove or disprove the following statement. It is possible to factor A as $A = LU$ with L lower triangular and U upper triangular.

3. Let $\|\mathbf{x}\|$ denote a vector norm on R^n.
   (a) Define $\|A\|$, the corresponding matrix norm (defined on the set of $n \times n$ matrices).
   (b) Recall that for $\mathbf{x} \in R^n$, $\|\mathbf{x}\|_2 = (x_1^2 + \cdots + x_n^2)^{1/2}$. Let D be a diagonal matrix with non zero diagonal elements $d_1, d_2, \ldots, d_n$. Prove that the 2-norm of D is given by

$$\|D\|_2 = \max_{1 \le i \le n} |d_i|.$$

   (c) Determine $\text{cond}_2(D)$ in terms of the d_i.

4. Suppose we are given $f(0) = 1.0, f(0.5) = 1.6, f(1) = 1.8$. Also, on $[0,1], |f'(x)| \le 2, |f''(x)| \le 3, |f'''(x)| \le 5, |f''''(x)| \le 8$.
   (a) What is the best estimate of $f(0.4)$?
   (b) What is the best estimate of $\int_0^1 f(x)\,dx$?
   (c) What is the best estimate of $f'(0.5)$?
   In each case bound the error.

5. Find constants $\alpha_1, \alpha_2, \alpha_3$ such that the integration rule

$$\int_0^1 f(x)\,dx \approx \alpha_1 f(1/4) + \alpha_2 f(1/2) + \alpha_3 f(3/4)$$

is exact for all polynomials of degree ≤ 2. For which polynomials will this rule give the correct answer ?

6. The function $f(x) = x^3 + x - 11$ has a root α which lies between $x = 2$ and $x = 3$.
   (a) Will the iteration scheme

$$x_{n+1} = 11 - x_n^3, \ n \geq 0, \ x_0 = 2$$

   converge to α ? Explain.

   (b) Find a number a so that the iteration scheme

$$x_{n+1} = \frac{11 - x_n^3 + a x_n}{1 + a}, \ n \geq 0, \ x_0 = 2$$

   will converge to α. Find x_1 and x_2 using your value of a.

   (c) Set up Newton's method for solving $f(x) = 0$ and use it to find two new approximations x_1 and x_2, starting at $x_0 = 2$.

   (d) Let $x_0 = 2, x_1 = 3$. Use the secant method to find a new approximation x_2.

7. We wish to solve the system

$$x^2 y + y^2 - x = 2, \quad 3x + y - x^2 y^2 = 2$$

   by Newton's method. If $(x_0, y_0) = (1, 1)$ what is (x_1, y_1) ? Do you think that (x_1, y_1) is closer to a root than (x_0, y_0) ?