

1. Read section 2.5 of *Lay* at least through the material in the box on p.146. Then do problem 31 on p.150. In part (a) use the command $[L,U]=lu(A)$. In part (b) use the backslash operator twice. (See equation (2) on p.143.)

MATLAB uses finite precision arithmetic, so there will be round-off error in most calculations. In order to minimize the effect of round-off error the Gaussian elimination process must be modified by introducing the concept of *partial pivoting*. (The definition is given on p.20 of *Lay*.) The MATLAB command $A \setminus \mathbf{b}$ implements Gaussian elimination with partial pivoting. What accuracy should one expect ?

MATLAB carries about 15 digits so its basic calculations are very accurate, with errors about 10^{-15} . We can quantify this more precisely. The numbers that can be represented in MATLAB are called floating point numbers. There is a special number in MATLAB denoted by *eps*, that is defined to be the distance from 1 to the next floating point number. The command `eps` will produce *eps* for the computer you are using; *eps* is approximately $2.22e - 16$. The significance of *eps* is that it determines the round-off error committed when a number is rounded: If a positive number a is rounded, the round-off error is approximately $a \times eps$.

Consider the linear system

$$A\mathbf{x} = \mathbf{b},$$

and let \mathbf{x}^* be the computed solution (the number produced by the computer). There are two measures of accuracy of the computed solution \mathbf{x}^* . The *error* in \mathbf{x}^* is defined to be

$$\mathbf{e} = \mathbf{x} - \mathbf{x}^*,$$

and the *residual* is defined to be

$$\mathbf{r} = \mathbf{b} - A\mathbf{x}^*.$$

The *norm* (size) of \mathbf{e} is one measure of the accuracy of \mathbf{x}^* ; if the norm of \mathbf{e} is small, then \mathbf{x}^* is close to \mathbf{x} . The second measure of accuracy of \mathbf{x}^* is the norm of \mathbf{r} ; if the norm of \mathbf{r} is small, then \mathbf{x}^* nearly satisfies the linear system.

There are two principles that indicate the accuracy of the solution \mathbf{x}^* produced by Gaussian elimination with partial pivoting (the method used in $A \setminus \mathbf{b}$). The first is that the norm of the residual is nearly always small; to be more precise,

$$\text{Norm of } \mathbf{r} \propto \text{Norm of } \mathbf{x} \times \text{Norm of } A \times eps,$$

i.e., for a moderate sized problem the norm of the residual is approximately *eps*. The second principle is that

$$\text{Norm of } \mathbf{e} \propto \text{Norm of } \mathbf{x} \times \text{cond}(A) \times eps,$$

where $\text{Cond}(A)$ is a number that measures how close A is to being singular. By the norm of a vector or a matrix we mean roughly speaking, the maximum of the absolute

values of the elements of the vector or the matrix, respectively. If A is nearly singular, then $\text{Cond}(A)$ is large, and the error may be large. Matrices for which $\text{cond}(A)$ is large are called *ill-conditioned*. If $\text{Cond}(A) \approx 10^t$, then the error would be about $10^t \times \text{eps} \approx 10^{16-t}$, i.e., we would have lost approximately t digits of accuracy. $\text{Cond}(A)$ can be found with the command **cond(A)**. When $A \setminus \mathbf{b}$ is used, if the calculations indicate that A is ill-conditioned, a warning message is printed, and a number RCOND which is an approximation to $1/\text{cond}(A)$, is printed. The following two problems illustrate these principles. Note: type **format long** before you do the requested calculations.

- The $n \times n$ matrix H_n whose elements are given by $h_{i,j} = 1/(i + j - 1)$ is very ill-conditioned if n is large; it is called the *Hilbert Matrix*. It can be generated in MATLAB with the command **hilb(n)**. Let A be the Hilbert matrix of size 11. Let $\mathbf{x} = \mathbf{ones}(11,1)$, and let $\mathbf{b} = A\mathbf{x}$. Now solve the system $A\mathbf{x} = \mathbf{b}$, obtaining \mathbf{x}^* . Since we know \mathbf{x} exactly we can compute \mathbf{e} . Do this and also compute \mathbf{r} . What are the norms of these vectors? Are the above principles satisfied for this example?

- Let

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 6 & 3 & -2 \\ 7 & 8 & 9 & -3 & 3 \\ 3 & 4 & -2 & -4 & 10 \\ 1 & 2 & 3 & 4 & 6 \end{pmatrix}.$$

Find $\text{cond}(A)$. Is A ill-conditioned? Solve $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b} = (55, 34, 53, 39, 60)^T$. How small would you expect \mathbf{e} and \mathbf{r} to be? The exact solution is $\mathbf{x} = (1, 2, 3, 4, 5)^T$. Calculate \mathbf{e} and \mathbf{r} . Are the above principles satisfied for this example?