CMSC/MAPL 460 SAMPLE FINAL EXAM

1.
   (a) For two-digit (base 10) arithmetic with rounding, what is the machine epsilon ?

   Consider the system

$$\begin{aligned} \frac{1}{4}x + \frac{1}{5}y &= \frac{3}{4} \\ \frac{1}{3}x + \frac{1}{4}y &= 1 \end{aligned} \tag{1}$$

   (b) Solve the system (1) using two-digit arithmetic (rounding to two significant figures after <u>every</u> arithmetic operation) and Gaussian elimination with partial pivoting. (Do not scale the equations.)
   (c) Solve the system (1) using exact arithmetic.

   If you are careful you should see a considerable differene betwen the solutions found in (b) and (c).

   (d) For the solution found in (b) compute the residual (using exact arithmetic).
   (e) Compute the condition number of the coefficient matrix in (1) in the infinity norm.
   (f) What lessons can be drawn from the above computations ?

2. Let $f(x) = \sqrt{x^2+1} - x$.
   (a) For which values of x is it difficult to compute $f(x)$ accurately on a computer ?
   (b) For these values of x find a good way to compute $f(x)$.

3.

$$\text{Let } A = \begin{pmatrix} 1 & 2 \\ 1 & 2.01 \end{pmatrix}, \text{ so that } A^{-1} = \begin{pmatrix} 201 & -200 \\ -100 & 100 \end{pmatrix}, \ \mathbf{b_1} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}, \ \mathbf{b_2} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

   (a) Find the condition number of A in the ∞-norm.
   (b) Solve $A\mathbf{x} = \mathbf{b_1}$ and $A\mathbf{x} = \mathbf{b_2}$. What important principle of numerical analysis do the results serve to illustrate ?

4. Suppose we use Simpson's rule with 400 panels to approximate

$$I = \int_0^1 (x^3 + 3x^2 - x)dx.$$

   Assuming no roundoff error, what will the result be ? Explain.

5. Compute an approximation T_4 to $I = \int_{-1}^{1} |x|\, dx$ using the trapezoid rule with 4 panels. Compute the error $I - T_4$. Explain you results.

6. Let $f(x) = 2x^3 + 2x - 1$.
   (a) Show that $f(x) = 0$ has one real solution, α with $0 < \alpha < 1$.

(b) Obtain an approximation to α in the following way: Find $p_2(x)$, the quadratic polynomial interpolating $f(x)$ at $x = 0, x = \frac{1}{2}$ and $x = 1$. Then find the root of $p_2(x) = 0$ lying in $[0, 1]$.

(c) Set up Newton's method for finding α and find two new approximations x_1 and x_2 starting with $x_0 = 0$.

(d) Let $x_0 = 0,\ x_1 = 1$. Use the secant method to find a new appoximation to α, x_2.

(e) Recast the problem of solving $f(x) = 0$ as a fixed point problem of the form $x = g(x)$ such that, given an appropriate starting value x_0, the iteration $x_{n+1} = g(x_n)$ converges to α. (Do not use the Newton iterates.) Show that your scheme works.

7. Let $\mathbf{f} : \mathbf{R^n} \to \mathbf{R^n}$ be a smooth function. What is Newton's method for solving $\mathbf{f}(\mathbf{x}) = \mathbf{0}$? Be sure to define any terms you use.

8. Let

$$g(x) = -4 + 4x - \frac{1}{2}x^2.$$

(a) Find the fixed points of g.

(b) Show that fixed point iterations can be used to find one fixed point but not the other. Explain by reference to a theorem why this is so.

9. We wish to fit the data (0,1), (1,3), (2,7), (3,10), (4,20) to a function of the form

$$f(x) = a + bx + ce^x$$

in the sense of least squares. Find an equation for the coefficients a, b and c. <u>Do not</u> do any computations.

10.

(a) Find the weights ω_0 and ω_1 and the node x_1 such that the integration rule

$$I = \int_0^h f(x)\,dx \approx \omega_0 f(0) + \omega_1 f(x_1) = Q$$

is exact for all quadratic polynomials.

(b) Is the above rule exact for cubics ?

(c) Given that the error of the rule is of the form

$$e = I - Q = cf^{(d)}(\zeta)h^{d+1}$$

for smooth f, where $0 < \zeta < h$, find the integer d and the constant c.

11. Consider the initial value problem

$$y'' + y' - 2y = 2t,\ y(0) = 0,\ y'(0) = 1 \tag{1}$$

(a) Transform (1) into an initial value problem for a first order system.

(b) Compute an approximation to the solution of (1) at $t = 0.2$ by using two steps of Euler's method with $h = 0.1$. Exact values are:

$$y(0.1) = 0.095806,\ y'(0.1) = 0.923902,\ y(0.2) = 0.186243,\ y'(0.2) = 0.891722.$$