CMSC/MAPL 460 Dr. Wolfe ASSIGNMENT #2 Due March 1, 2004

1. Ex. 2.1, part (b), p.42, *Shampine, Allen & Preuss.*
2. Ex. 2.5, part (b), p.48, *Shampine, Allen & Preuss.*
3. Consider the linear system

$$
\begin{aligned}
6x_1 + 2x_2 + 2x_3 &= -4 \\
2x_1 + \frac{2}{3}x_2 + \frac{1}{3}x_3 &= 2 \\
x_1 + 2x_2 - x_3 &= 0
\end{aligned}
$$

   (a) Verify that its solution is

$$x_1 = 5.2 \quad x_2 = -7.6 \quad x_3 = -10.0$$

   (b) Using four digit floating-point decimal arithmetic with rounding, solve the system by Gaussian elimination without pivoting.

   (c) Repeat part (b) using partial pivoting.

   In performing the arithmetic operations, remember to round to four significant digits after each operation, just as would be done on a computer.
   What do you conclude ?

4. The Hilbert matrix of order n, H_n is defined by

$$(H_n)_{i,j} = \frac{1}{i+j-1}, \quad i = 1, \ldots, n, \quad j = 1, \ldots, n.$$

   H_n is nonsingular. However, as n increases, the condition number of H_n increases rapidly. H_n is a library function in MATLAB, hilb(n). Let $n = 10, \mathbf{x} = \text{ones}(10,1)$ and $\mathbf{b} = H_{10}\mathbf{x}$. Now use the backslash operator to solve the system $H_n\mathbf{x} = \mathbf{b}$, obtaining $\mathbf{x}^*$. Since we know $\mathbf{x}$ exactly, we can compute $\mathbf{e} = \mathbf{x} - \mathbf{x}^*$, the error, and $\mathbf{r} = \mathbf{b} - H_{10}\mathbf{x}^*$, the residual. Compute these quantities and also cond(H_n) (a MATLAB function). Show that the two basic principles of solving linear systems by G.E./P.P. in floating point arithmetic hold. How many correct digits does $\mathbf{x}^*$ have ? Repeat with $n = 11, 12, \ldots$. Stop when some component of $\mathbf{x}^*$ has no correct digits.

5. Define the matrix A_n of order n by

$$A_n = \begin{bmatrix} 1 & -1 & -1 & -1 & \cdots & -1 \\ 0 & 1 & -1 & -1 & \cdots & -1 \\ & & & \ddots & & \\ \vdots & & & & 1 & -1 \\ 0 & & \cdots & & 0 & 1 \end{bmatrix}$$

   (a) Find the inverse of A_n explicitly.

Hint: Find the inverse of A_6 by using MATLAB. Then use the result to "guess" the inverse of A_n in general.

(b) Calculate $\text{cond}(A_n)$ in the ∞-norm.

(c) With $\mathbf{b} = [-n+2, -n+3, \ldots, -1, 0, 1]^T$, the solution of $A_n\mathbf{x} = \mathbf{b}$ is $\mathbf{x} = [1, 1, \ldots, 1]^T$. Perturb $\mathbf{b}$ to $\hat{\mathbf{b}} = \mathbf{b} + [0, \ldots, 0, \epsilon]^T$. Solve for $\hat{\mathbf{x}}$ in $A_n\hat{\mathbf{x}} = \hat{\mathbf{b}}$. Show that these values of $\mathbf{b}, \hat{\mathbf{b}}, \mathbf{x}, \hat{\mathbf{x}}$ satisfy the fundamental inequality,

$$\frac{\|\mathbf{x} - \hat{\mathbf{x}}\|}{\|\mathbf{x}\|} \le \text{cond}(A_n)\frac{\|\mathbf{b} - \hat{\mathbf{b}}\|}{\|\mathbf{b}\|}.$$

(Use the ∞-norm.)
Hint $\hat{\mathbf{x}} = \mathbf{x} + A_n^{-1}(\hat{\mathbf{b}} - \mathbf{b})$.

6. Suppose $\mathbf{x}$ satisfies $A\mathbf{x} = \mathbf{b}$ and $\mathbf{x} + \mathbf{\Delta x}$ satisfies $(A + \Delta A)(\mathbf{x} + \mathbf{\Delta x}) = \mathbf{b} + \mathbf{\Delta b}$. Then we have the *condition number inequality*: If $\rho = \|A^{-1}\| \cdot \|\Delta A\| < 1$

$$\frac{\|\mathbf{\Delta x}\|}{\|\mathbf{x}\|} \le \frac{\text{cond}(A)}{1-\rho}\left(\frac{\|\Delta A\|}{\|A\|} + \frac{\|\mathbf{\Delta b}\|}{\|\mathbf{b}\|}\right). \qquad (1)$$

Consider the linear system $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{pmatrix} .5055 & .6412 & .8035 \\ .1693 & .0162 & .6978 \\ .5280 & .8369 & .4617 \end{pmatrix}, \quad \mathbf{b} = \begin{bmatrix} .4939 \\ .4175 \\ .2923 \end{bmatrix}$$

(a) Solve $A\mathbf{x} = \mathbf{b}$ using the backslash operator.

(b) Use equation (1) to answer the following question: If each entry in A and $\mathbf{b}$ might have an error of $\pm.00005$, how reliable is $\mathbf{x}$? Use the ∞-norm.

(c) Let

$$\Delta A = .0001*\text{rand}(3) - .00005*\text{ones}(3), \; \Delta b = .0001*\text{rand}(3,1) - .00005*\text{ones}(3,1).$$

Solve $(A + \Delta A)(\mathbf{x} + \mathbf{\Delta x}) = \mathbf{b} + \mathbf{\Delta b}$ to get $\mathbf{x} + \mathbf{\Delta x}$. Calculate $\|\mathbf{\Delta x}\|/\|\mathbf{x}\|$. Is this consistent with (b) ? What is the relative change in each x_i ?

7.

(a) Let A be an $n \times n$ matrix and $\mathbf{x} \in \mathbf{R}^n$. How many *flops* does it take to form the product $A\mathbf{x}$?

(b) Let A and B be $n \times n$ matrices. How many *flops* does it take to form the product AB ?

(c) In light of the results of (a) and (b), from the standpoint of efficiency, how should one compute $A^k\mathbf{x}$ for k a positive integer $k > 1$?

8. Construct a tridiagonal solver along the lines outlined on pp.68-69 of *Shampine, Allen & Preuss.* Your solver should be a function M-file with four vectors as input and the solution as the output. Test your solver on the problem $A\mathbf{x} = \mathbf{b}$ where A is the 9×9 tridiagonal matrix with 2's on the main diagonal and -1's on the super- and subdiagonals and $\mathbf{b}(j) = .01$, $j = 1, \ldots, 9$. As a check, the answer should be $\mathbf{x}(j) = \frac{1}{2}(\frac{j}{10})(1 - \frac{j}{10})$.