AMSC/CMSC 460 Sample Final Exam

1.

(a) Compute

$$\lim_{x\to 0}\frac{1-\cos x}{x^2}.$$

(b) Let $g(x) = \frac{1-\cos x}{x^2}$. Write a MATLAB script which produces a column vector x of length 10 whose i^{th} component is $g(10^{-i})$.

(c) Here is the result of running the script produced in part (ii):

$$x = \begin{matrix} 0.49958347219743 \\ 0.49999583334737 \\ 0.49999995832550 \\ 0.49999999696126 \\ 0.50000004137019 \\ 0.50004445029117 \\ 0.49960036108132 \\ 0 \\ 0 \\ 0 \end{matrix}$$

How do you account for this result ?

2.

(a) Find the LU factorization of

$$A = \begin{pmatrix} 2 & -1 & 3 \\ -1 & -2 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

(b) Use the factorization to solve $A\mathbf{x} = \mathbf{b}$ where $\mathbf{b} = (1, 2, 3)^T$.

3.

(a) Find the weights ω_0 and ω_1 and the node x_1 such that the integration rule

$$I = \int_0^h f(x)\,dx \approx \omega_0 f(0) + \omega_1 f(x_1) = Q$$

is exact for all quadratic polynomials.

(b) Is the above rule exact for cubics ?

(c) Given that the error of the rule is of the form

$$e = I - Q = cf^{(d)}(\zeta)h^{d+1}$$

for smooth f, where $0 < \zeta < h$, find the integer d and the constant c.

4. Let $f(x) = 2x^3 + 2x - 1$.
   (a) Show that $f(x) = 0$ has one real solution, α with $0 < \alpha < 1$.
   (b) Obtain an approximation to α in the following way: Find $p_2(x)$, the quadratic polynomial interpolating $f(x)$ at $x = 0, x = \frac{1}{2}$ and $x = 1$. Then find the root of $p_2(x) = 0$ lying in $[0, 1]$.
   (c) Set up Newton's method for finding α and find two new approximations x_1 and x_2 starting with $x_0 = 0$.
   (d) Let $x_0 = 0,\ x_1 = 1$. Use the secant method to find a new appoximation to α, x_2.
   (e) Recast the problem of solving $f(x) = 0$ as a fixed point problem of the form $x = g(x)$ such that, given an appropriate starting value x_0, the iteration $x_{n+1} = g(x_n)$ converges to α. (Do not use the Newton iterates.) Show that your scheme works

5. Consider the initial value problem

$$y'' + y' - 2y = 2t,\ y(0) = 0,\ y'(0) = 1 \tag{1}$$

   (a) Transform (1) into an inital value problem for a first order system.
   (b) Compute an approximation to the solution of (1) at $t = 0.2$ by using two steps of Euler's method with $h = 0.1$. Exact values are:

$$y(0.1) = 0.095806,\ y'(0.1) = 0.923902,\ y(0.2) = 0.186243,\ y'(0.2) = 0.891722$$