CMSC/MAPL 460 Dr. Wolfe ASSIGNMENT #2 Due March 5, 2001

1. Ex. 2.1, parts (a),(d), p.42, *Shampine, Allen & Preuss.*
2. Ex. 2.5, part (a), p.48, *Shampine, Allen & Preuss.*
3. Ex. 2.6, p.48, *Shampine, Allen & Preuss.*
4. Consider the linear system

$$\begin{aligned} 6x_1 + 2x_2 + 2x_3 &= -2 \\ 2x_1 + \frac{2}{3}x_2 + \frac{1}{3}x_3 &= 1 \\ x_1 + 2x_2 - x_3 &= 0 \end{aligned}$$

   (a) Verify that its solution is

$$x_1 = 2.6 \quad x_2 = -3.8 \quad x_3 = -5.0$$

   (b) Using four digit floating-point decimal arithmetic with rounding, solve the system by Gaussian elimination without pivoting.

   (c) Repeat part (b) using partial pivoting. In performing the arithmetic operations, remember to round to four significant digits after each operation, just as would be done on a computer. If you are careful you should see a significant difference.

5. The Hilbert matrix of order $n$, $H_n$ is defined by

$$(H_n)_{i,j} = \frac{1}{i+j-1}, \quad i = 1, \ldots, n, \quad j = 1, \ldots, n.$$

   $H_n$ is nonsingular. However, as $n$ increases, the condition number of $H_n$ increases rapidly. $H_n$ is a library function in MATLAB, hilb($n$). Let $n = 10$, $\mathbf{x}$=ones(10,1) and $\mathbf{b} = A\mathbf{x}$. Now use the backslash operator to solve the system $A\mathbf{x} = \mathbf{b}$, obtaining $\mathbf{x}^*$. Since we know $\mathbf{x}$ exactly, we can compute $\mathbf{e} = \mathbf{x} - \mathbf{x}^*$, the error, and $\mathbf{r} = \mathbf{b} - H_{10}\mathbf{x}^*$, the residual. Compute these quantities and also cond($A$) (a MATLAB function). Show that the two basic principles of solving linear systems by G.E./P.P. in floating point arithmetic hold. Repeat with $n = 15$.

6. Define the matrix $A_n$ of order $n$ by

$$A_n = \begin{bmatrix} 1 & -1 & -1 & -1 & \cdots & -1 \\ 0 & 1 & -1 & -1 & \cdots & -1 \\ & & & \ddots & & \\ \vdots & & & & 1 & -1 \\ 0 & & \cdots & & 0 & 1 \end{bmatrix}$$

   (a) Find the inverse of $A_n$ explicitly.

Hint: Find the inverse of $A_6$ by using MATLAB. Then use the result to "guess" the inverse of $A_n$ in general.

(b) Calculate $\text{cond}(A_n)$ in the $\infty$-norm.

(c) With $\mathbf{b} = [-n+2, -n+3, \ldots, -1, 0, 1]^T$, the solution of $A_n\mathbf{x} = \mathbf{b}$ is $\mathbf{x} = [1, 1, \ldots, 1]^T$. Perturb $\mathbf{b}$ to $\hat{\mathbf{b}} = \mathbf{b} + [0, \ldots, 0, \epsilon]^T$. Solve for $\hat{\mathbf{x}}$ in $A_n\hat{\mathbf{x}} = \hat{\mathbf{b}}$. Show that these values of $\mathbf{b}, \hat{\mathbf{b}}, \mathbf{x}, \hat{\mathbf{x}}$ satisfy the fundamental inequality,

$$\frac{\|\mathbf{x} - \hat{\mathbf{x}}\|}{\|\mathbf{x}\|} \leq \text{cond}(A_n)\frac{\|\mathbf{b} - \hat{\mathbf{b}}\|}{\|\mathbf{b}\|}.$$

(Use the $\infty$-norm.)
Hint $\hat{\mathbf{x}} = \mathbf{x} + A_n^{-1}(\hat{\mathbf{b}} - \mathbf{b})$.

7. Suppose $\mathbf{x}$ satisfies $A\mathbf{x} = \mathbf{b}$ and $\mathbf{x} + \mathbf{\Delta x}$ satisfies $(A + \Delta A)(\mathbf{x} + \mathbf{\Delta x}) = \mathbf{b} + \mathbf{\Delta b}$. Then we have the *condition number inequality*: If $\rho = \|A^{-1}\|\|\Delta A\| < 1$

$$\frac{\|\mathbf{\Delta x}\|}{\|\mathbf{x}\|} \leq \frac{\text{cond}(A)}{1-\rho}\left(\frac{\|\Delta A\|}{\|A\|} + \frac{\|\mathbf{\Delta b}\|}{\|\mathbf{b}\|}\right). \tag{1}$$

Consider the linear system $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{pmatrix} .5055 & .6412 & .8035 \\ .1693 & .0162 & .6978 \\ .5280 & .8369 & .4617 \end{pmatrix}, \quad \mathbf{b} = \begin{bmatrix} .4939 \\ .4175 \\ .2923 \end{bmatrix}$$

(a) Solve $A\mathbf{x} = \mathbf{b}$ using the backslash operator.

(b) Use equation (1) to answer the following question: If each entry in $A$ and $\mathbf{b}$ might have an error of $\pm.00005$, how reliable is $\mathbf{x}$ ? Use the $\infty$-norm.

(c) Let

$$\Delta A = .0001{*}\text{rand}(3) - .00005{*}\text{ones}(3),\ \Delta b = .0001{*}\text{rand}(3,1) - .00005{*}\text{ones}(3,1).$$

Solve $(A + \Delta A)(\mathbf{x} + \mathbf{\Delta x}) = \mathbf{b} + \mathbf{\Delta b}$ to get $\mathbf{x} + \mathbf{\Delta x}$. Calculate $\|\mathbf{\Delta x}\|/\|\mathbf{x}\|$. Is this consistent with (b) ?

8.

(a) Let $A$ be an $n \times n$ matrix and $\mathbf{x} \in \mathbf{R^n}$. How many *flops* does it take to form the product $A\mathbf{x}$ ?

(b) Let $A$ and $B$ be $n \times n$ matrices. How many *flops* does it take to form the product $AB$ ?

(c) In light of the results of (a) and (b), from the standpoint of efficiency, how should one compute $A^k\mathbf{x}$ for $k$ a positive integer, $k > 1$ ?