AMSC/CMSC 460 SUMMER 2004

SAMPLE MIDTERM EXAM

1. Let N be a positive integer. Consider the following MATLAB script:

$$
\begin{array}{l}
y = 0; \\
\text{for } i = 1 : N \\
y = y + (1/N); \\
\text{end} \\
y
\end{array}
$$

   (a) What would the result of the computation be in exact arithmetic ?
   (b) When the script was actually run with $N = 100,000 (= 10^5)$ the result was $y = .99999999999808$. When it was run with $N = 131,072 (= 2^{17})$ the result was $y = 1$. Explain these results.

2. Let

$$A = \begin{pmatrix} 4 & 6 \\ 6 & 13 \end{pmatrix}.$$

   A is symmetric, positive definite.
   (a) Find a lower triangular matrix L with positive diagonal entries such that $A = LL^T$ (Cholesky factorization).
   (b) Let $\mathbf{b} = (-2, 1)^T$. Use the factorization of part (a) to solve $A\mathbf{x} = \mathbf{b}$ by forward elimination and back substitution.

3. Let $f(x) = x^3$.
   (a) Find the quadratic polynomial $p_2(x)$ interpolating $f(x)$ at $x_0 = 0$, $x_1 = 1$, $x_2 = 2$.
   (b) Find $P(x)$, the piecewise linear interpolent to $f(x)$ with breakpoints x_0, x_1, x_2.
   (c) Find the linear function $L(x)$ which best fits the data $(0,0), (1,1), (2,8)$ in the sense of least squares. Plot $f(x)$ and $L(x)$ on the same graph.
   (d) Which of the functions found in parts (a),(b), and (c) do you think does the best job of approximating $f(x)$ on $[0,2]$? Explain.

4. Let

$$S(x) = \begin{cases} x + 1 & -2 \le x \le -1, \\ x^3 - 2x - 1 & -1 \le x \le 1, \\ x - 3 & 1 \le x \le 2 \end{cases}$$

   Is $S(x)$ a natural cubic spline ? Explain.

5.
   (a) Find a polynomial $p(x)$ of degree ≤ 2 satisfying $p(0) = 0$, $p(1) = 1$, $p'(1/4) = 2$.
   (b) There is a number c, $0 < c < 1$, such that there is no polynomial $p(x)$ of degree ≤ 2 satisfying $p(0) = 0$, $p(1) = 1$, $p'(c) = 2$. Find c.

6. Let

$$I = \int_0^1 \frac{2}{1+x}\, dx = \ln 4 = 1.38629436$$

Compute approximations to I using

(a) The 4 panel trapezoid rule.

(b) The 4 panel Simpson's rule.

(c) The 4 panel corrected trapezoid rule.

Which method gives the best result ?

7. Mark each of the following statements as true (T) or false (F).

(a) A problem is ill-conditioned if its solution is highly sensitive to small changes in the problem data.

(b) If x is any vector in $\mathbf{R}^n$ then $\|x\|_1 \geq \|x\|_\infty$.

(c) Any nonsingular matrix A can be factored as $A = LU$ with L lower triangular and U upper triangular.

(d) When interpolating a continuous function by polynomials at equally spaced points on a given interval, the polynomial interpolants always converge to the function as the number of interpolation points increases.

(e) Given $n+1$ data points $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$ with $x_0 < x_1 < \cdots < x_n$ there is a unique cubic spline interpolating these points.