MATH 436 HOMEWORK 2 SOLUTIONS

These solutions are meant to be a grading rubric for me. They are not necessarily the most detailed or perfectly accurate. Please let me know if you encounter any mistakes.

1. Problem 1

a.) Recall that we can parametrize our torus as

$$\mathbb{T} := \{([a + r\cos(y)]\cos(x), [a + r\cos(y)]\sin(x), r\sin(y)) \mid 0 \leq x \leq 2\pi,\ -\pi \leq y \leq \pi\}.$$

Consider the mapping associated to this parametrization:

$$F : \mathbb{R}^2 \to \mathbb{R}^3,\ F(x, y) := ([a + r\cos(y)]\cos(x), [a + r\cos(y)]\sin(x), r\sin(y))$$

Note that F is 2π-periodic. One possible set of charts is:

(1) $U_1 = F((-\frac{\pi}{2}, \frac{3\pi}{2}) \times (-\frac{\pi}{2}, \frac{3\pi}{2}))$, with mapping $\phi_1(x, y, z) = (\alpha, \beta)$; here $\alpha = \arctan(\frac{y}{x})$ if $x > 0$, $\frac{\pi}{2}$ if $x = 0, y > 0$, and $\arctan(\frac{y}{x}) + \pi$ is $x < 0$; $\beta = \arcsin(\frac{z}{r})$ if $x^2 + y^2 \geq a^2$ and $\arcsin(\frac{z}{r}) + \pi$ if $x^2 + y^2 < a^2$.

(2) $U_2 = \{(x, y, z) \in \mathbb{T}\ |z^2 = a^2 - (r - \sqrt{x^2 + y^2})^2, z > 0\}, U_3 = \{(x, y, z) \in \mathbb{T}\ |z^2 = a^2 - (r - \sqrt{x^2 + y^2})^2, z < 0\}$, both with map $\phi_2(x, y, z) = \phi_3(x, y, z) = (x, y)$.

(3) $U_4 = \{(x, y, z) \in \mathbb{T}\ |x^2 = r - (r - \sqrt{a^2 - z^2})^2 - y^2, x > 0\}, U_5 = \{(x, y, z) \in \mathbb{T}\ |x^2 = r + (r - \sqrt{a^2 - z^2})^2 - y^2, x > 0\}$ with maps $\phi_4(x, y, z) = \phi_5(x, y, z) = (y, z)$.

2.) One pair of intersecting charts is U_3 and U_5. A computation yields $\phi_5 \circ \phi_3^{-1}$ and $\phi_3 \circ \phi_5^{-1}$ are inverses, as hoped. Indeed:

$$\phi_5 \circ \phi_3^{-1}(s, t) = \left(t, \sqrt{a^2 - (r - \sqrt{s^2 + t^2})^2}),\ \phi_3 \circ \phi_5^{-1}(s, t) = (\sqrt{(r + \sqrt{a^2 - t^2})^2 - s^2}, s\right).$$

The result follows by a calculation.

3.) We perform the calculation only for the transition map $\phi_5 \circ \phi_3^{-1}(s, t) = \left(t, \sqrt{a^2 - (r - \sqrt{s^2 + t^2})^2}\right)$; in this case:

$$D\phi_5 \circ \phi_3^{-1} = \frac{1}{\sqrt{(s^2+t^2)(a^2 - (r - \sqrt{s^2+t^2})^2)}} \begin{bmatrix} 0 & \sqrt{s^2 + t^2}\sqrt{a^2 - (r - \sqrt{s^2 + t^2})^2} \\ s(r - \sqrt{s^2 + t^2}) & t(r - \sqrt{s^2 + t^2}) \end{bmatrix}$$

2. Problem 2

To start, we notice that det is a multilinear alternating tensor of degree n. Moreover, the space of such tensors, $\Lambda^n T_pM$ is seen to be 1 dimensional by a counting arguments. It is not hard to show from this that $\det = dx^1 \wedge dx^2 \wedge ... \wedge dx^n$. Now, recall that det is a multiplicative homomorphism on the space of invertible matrices, that is, $det(AB) = det(A)det(B)$. Let E_i, $i = 1, ..., n$ be an orthonormal basis for T_pM; write in coordinates $E_i = \sum_j a_i^j \frac{\partial}{\partial x_j}$. For notational ease with applying the determinant operator, let us denote $(E_1, ..., E_n)$ simply as E. Notice that since $g(E_i, E_j) = \delta_{ij}$, we have $\sum_{k,\ell} g_{ij} a_i^k a_j^\ell = \delta_{ij}$. By definition of matrix multiplication, $EgE^T = I$. We compute:

$$
\begin{aligned}
&\sqrt{\det(g_{ij}(x))} dx^1 \wedge ... \wedge dx^n (E_1, ..., E_n) \\
=&\sqrt{\det(g_{ij}(x))}\,\det(E) \\
=&sgn(\det(E))\sqrt{\det(g_{ij}(x))\det(E)^2} \\
=&sgn(\det(E))\sqrt{\det(E^T g_{ij}(x) E)} \\
=&sgn(\det(E))\sqrt{\det(I)} \\
=&sgn(\det(E)).
\end{aligned}
$$

By definition of dV_g, the result is proven.