MATH 436 HOMEWORK 3 SOLUTIONS

These solutions are meant to be a grading rubric for me. They are not necessarily the most detailed or perfectly accurate. Please let me know if you encounter any mistakes.

1. Problem 1

For this problem, recall that $dx_j(\frac{\partial}{\partial x_i}) = \delta_{i,j} = 1$ if $i = j$, 0 otherwise.

a.) $dF(\frac{\partial}{\partial x_1}) = (2x_1, e^{x_1})$
b.) $dF(\frac{\partial}{\partial x_1}) = 0$; $dF(\frac{\partial}{\partial x_2}) = 1$.
c.) $dF = (3x_1^2 dx_1, 3x_2^2 dx_2, 3x_3^2 dx_3, x_2x_3dx_1 + x_1x_3dx_2 + x_1x_2dx_3)$.
Some students wrote this as a 4x3 matrix, which I gave credit for as well.
d.) Yes, the map $F : \mathbb{R}^3 \to \mathbb{R}^3$ given by $F(x_1, x_2, x_3) = (x_1+x_2, x_2+x_3, x_1+x_3)$ is a diffeomorphism. Notice that there is an associated matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

with the property that $F(x_1, x_2, x_3) = A(x_1, x_2, x_3)$, where the multiplication on the right hand side is that of a matrix on a vector. By a standard result, F is a diffeomorphism if this matrix is non-singular. The determinant of this matrix is computed to be $\det(A) = 2 \neq 0$, so we conclude F is a diffeomorphism.

2. Problem 2

a.) $dF(dx_1^2 + dx_2^2) = (4x_1^2 + e^{2x_1})dx_1^2$
b.) $dF(dx_2^2) = dx_2^2$; so, the induced metric on $\mathbb{R}^2$ only considers the first second coordinate.
c.) $dF(dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2) = 9x_1^2dx_1^2 + 9x_2^2dx_2^2 + 9x_3^2dx_3^2 + x_2^2x_3^2dx_1^2 + 2x_1x_2x_3^2dx_1dx_2 + 2x_1x_2^2x_3dx_1dx_3 + x_1^2x_3^2dx_2^2 + 2x_1^2x_2x_3dx_2dx_3 + x_1^2x_2^2dx_3^2$.
d.) $dF(dx_1^2 + dx_2^2 + dx_3^2) = dx_1^2 + dx_2^2 + dx_3^2$. This linear map preserves the metric.

3. Problem 3

Write $\gamma(t) = (\gamma_1(t), \gamma_2(t))$. We compute:

$$
\begin{aligned}
L(\gamma) &= \int_0^1 \sqrt{h(\dot\gamma(t), \dot\gamma(t))}dt \\
&= \int_0^1 \sqrt{h\left(\dot\gamma_1(t)\frac{\partial}{\partial x_1} + \dot\gamma_2(t)\frac{\partial}{\partial x_2}, \dot\gamma_1(t)\frac{\partial}{\partial x_1} + \dot\gamma_2(t)\frac{\partial}{\partial x_2}\right)} \\
&= \int_0^1 |\dot\gamma_2(t)|dt \\
&= \int_0^1 \dot\gamma_2(t)dt \\
&= \gamma_2(1) - \gamma_2(0).
\end{aligned}
$$

4. Problem 4

a.) We may write $dy^i = \sum_{k=1}^n \frac{\partial x_k}{\partial y_i} dx^k$, as per the classical change of variables formula. In terms of matrices, this can be written as

$$dy^i = \sum_{k=1}^n J_{k,i} dx^k,$$

where J is the Jacobian matrix for the change of variables $x \mapsto y$.

b.)

$$
\begin{aligned}
g_{i,j}(y(q)) &= g(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j}) \\
&= g(\sum_{k=1}^n J^{-1}_{k,i}\frac{\partial}{\partial x_k}, \sum_{\ell=1}^n J^{-1}_{k,j}\frac{\partial}{\partial x_\ell}) \\
&= \sum_{k,\ell=1}^k J^{-1}_{k,i} g_{k,\ell}(x(q)) J^{-1}_{\ell,j}.
\end{aligned}
$$

c.) The result follows directly by substitution:

$$
\begin{aligned}
\sqrt{\det[g_{i,j}(y(q))]}dy^1 \wedge dy^2 \wedge ... \wedge dy^n &= \left(\sqrt{\det[\sum_{k,\ell=1}^k J^{-1}_{k,i} g_{k,\ell}(x(q)) J^{-1}_{\ell,j}}\right) \det(J) dx^1 \wedge ... dx^n \\
&= \sqrt{\det g_{i,j}(x(q))}(\det(J^{-1})^T) \det(J^{-1}))^{\frac{1}{2}} \det(J) dx^1 \wedge ... \wedge dx^n \\
&= \sqrt{\det g_{i,j}(x(q))} dx^1 \wedge ... \wedge dx^n
\end{aligned}
$$