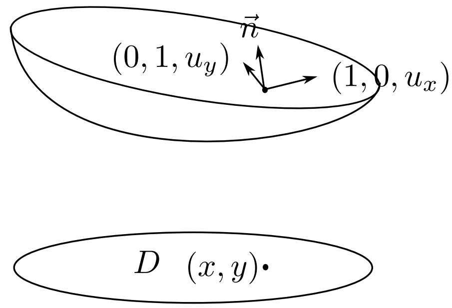
Lecture 1: Overview. The subdifferential of a convex function. The real Monge-Ampére operator - definition

There are many ways to introduce Monge-Ampère equations, including optimal transport as in previous classes. In the following we will introduce it by considering geometry of surfaces in $\mathbb{R}^3$. The simplest type of surfaces in $\mathbb{R}^3$ are graph of a function above a domain $D \subset \mathbb{R}^2$:

$$\mathrm{gr}(u)\Big|_D = \{(x, y, u(x, y))|(x, y) \in D\},$$



Gaussian curvature can be computed by the following recipe: the unit normal can be computed by crossing tangent vectors

$$\left(1, 0, \frac{\partial u}{\partial x}\right), \quad \left(0, 1, \frac{\partial u}{\partial y}\right)$$

and normalizing, which gives at point $(x, y, u(x, y))$

$$\vec{n} = \frac{\left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, -1\right)}{\sqrt{1 + |\nabla u|^2}}.$$

$\vec{n}$ can be viewed as a map to the unit sphere $\vec{n} : D \to S^2$ and the Gauss map is its differential:

$$G = d\vec{n} : D \to TS^2$$

Gaussian curvature is then defined as

$$K := \det d\vec{n} = \frac{u_{xx}u_{yy} - u_{xy}^2}{(1 + |\nabla u|^2)^2}$$

where K stands for *Krümmung* (German for curvature). Thus we have

$$\det \nabla^2 u = K(x, y)(1 + |\nabla u|^2)^2.$$

A **Monge-Ampère type equation** is the fully nonlinear PDE of the form

$$\det \nabla^2 u = F(\nabla u(\vec{x}), u(\vec{x}), \vec{x})$$

in $\mathbb{R}^n$. The case where $F = 0$ in $n = 2$ corresponds to $K = 0$, i.e., the graph of solution will be a flat surface above the interior of domain D.

Weak Solution

The idea to define weak solution is to replace the equation of determinant of Hessian by equation of a measure, i.e., for C^2 function u, we replace $\det \nabla^2 u$ by the measure $\det \nabla^2 u dx$ where $dx = dx^1 \wedge \ldots \wedge dx^n$. Observe that we have for any C^2 function u in domain E,

$$\int_E \det \nabla^2 u dx = \int_{\nabla u(E)} dy = \mathrm{meas}(\nabla u(E)),$$

since $\mathrm{Jac}(\nabla u) = \det \nabla^2 u$, that is, we really only need information about gradient ∇u. This inspires us to define

Definition For $u \in C^1$, MA(u) is a Boreal measure given by

$$\mathrm{MA}(u)(E) = \text{Lebesgue measure of } \nabla u(E)$$

for Borel set E.

Remark It is easy to check that this is indeed a Boreal measure, in particular, for any Borel set E the set $\nabla u(E)$ is indeed a Borel set since ∇u is continuous for $u \in C^1$. On the other hand note that

$$\mathrm{MA}(u)(E) = \int_E \det \nabla^2 u dx$$

for $u \in C^2$, therefore $\mathrm{MA}(u)$ is indeed a generalization of measure $\det \nabla^2 u dx$.

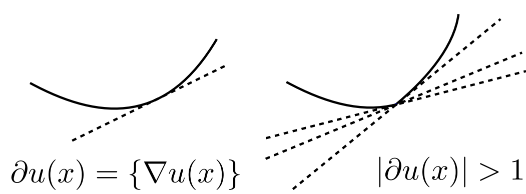
Now what if $u \notin C^1$? Actually we can generalize Monge-Ampère measure to convex functions, this is done by generalizing the notion of gradient:

Definition For u a finite convex function,

$$\partial u(x) := \{y \in \mathbb{R}^n | u(z) \geq u(x) + \langle y, z - x \rangle \ \forall z\}$$

is called the **subdifferential** of u at x and is in fact a convex set.

Remark The subdifferential of u can be understood as set of supporting planes. It is a singleton if u is differentiable at x, consist of just the tangent plane at x while when u is not differentiable there can be more than one supporting plane:



Definition For u convex,

$$\mathrm{MA}(u)(E) = \text{Lebesgue measure of } \partial u(E)$$

is called **Monge-Ampère measure** of E with respect to u

We will show $\mathrm{MA}(u)$ is a nonnegative Borel measure

Claim $\mathscr{A} := \{E \subseteq D : \partial u(E) \text{ is Lebesgue measurable}\}$ is a σ-algebra containing the Borel σ-algebra.

Proof will be given next time.

Lecture 2: The real Monge-Ampere operator - construction as a Borel measure using Alexandrov's theorem. Solution of the Dirichlet problem for the homogeneous real Monge-Ampere equation using an upper envelope

Recall $\mathrm{MA}(u)(E) :=$ Lebesgue measure of $\partial u(E)$ with $u \in \mathrm{Cvx}(D)$ for any Borel set E.

Theorem For convex function u, the Monge–Ampère measure $\mathrm{MA}(u)$ is well-defined, in other words, $\mathrm{MA}(u)$ is a Borel measure.

Theorem is a consequence of the following Claims:

Claim 1

$$\mathcal{A} := \{E \subseteq D : \partial u(E) \text{ is Lebesgue measurable }\}$$

is a σ-algebra containing the Borel σ-algebra.

Claim 2 For any $x_1 \neq x_2$

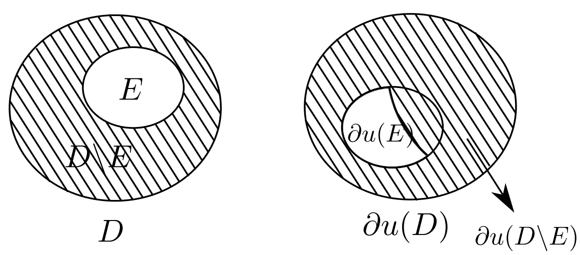
$$B := \{y \in \mathbb{R}^n : y \in \partial u(x_1) \cap \partial u(x_2), x_1 \neq x_2\}$$

is Lebesgue measurable and is in fact an Lebesgue null set. (Note that only proving $\partial u(x_1) \cap \partial u(x_2)$ for fixed $x_1 \neq x_2$ is null set is not enough for subsequent steps!)

Proof of Claim 1: (1) Compact sets are in $\mathcal{A}$: Let K be compact, $\partial u(K) = \bigcup_{x \in K} \partial u(x)$, let $\{y_i\} \subset \partial u(K)$ and $y_i \to y$, then $y_i \in \partial u(x_i)$ for some x_i, there is a convergent subsequence, thus without loss of generality assume $x_i \to x$ and thus for all z, $u(z) \geq u(x_i) + \langle z, y_i \rangle \to u(x) + \langle z, y \rangle$ therefore $y \in \partial u(x) \subset \partial u(K)$ hence $\partial u(K)$ is closed thus Lebesgue measurable.

(2) For $E \in \mathcal{A}$, the complement $D \backslash E \in \mathcal{A}$. One has

$$\partial u(D \backslash E) = (\partial u(D) \backslash \partial u(E)) \cup (\partial u(D \backslash E) \cap \partial u(E))$$



where by Claim 2 $\partial u(D \backslash E) \cap \partial u(E)$ is a Lebesgue null set, and the other two sets ($\partial u(D)$ and $\partial u(E)$) are already Lebesgue measurable, the conclusion follows.

(3) $\mathcal{A}$ is closed under countable unions: it suffices to show closedness under union, let $E_1, E_2 \in \mathcal{A}$,

$$\partial u(E_1 \cup E_2) = \partial u(E_1) \cup \partial u(E_2)$$

is measurable since it is union of two Lebesgue measurable sets.

(4) $\mathcal{A}$ is closed under countable intersection. Let $E_1, E_2 \in \mathcal{A}$, it is easy to check that

$$\partial u(E_1 \cap E_2) = (\partial u(E_1) \cap \partial u(E_2)) \backslash (\partial u(E_1 \backslash E_2) \cap \partial u(E_2 \backslash E_1))$$

where the last set $\partial u(E_1 \backslash E_2) \cap \partial u(E_2 \backslash E_1)$ is null set by Claim 2, the conclusion follows. $\square$

Proof of Claim 2: (1) Any convex function u over D is Lipschitz over $\forall \Gamma \Subset D$, i.e. $\exists C = C(\Gamma, D)$ such that

$$\|u\|_{C^{0,1}(\Gamma)} \leq C$$

Proof: By (1) of proof of Claim 1, $\partial u(\Gamma)$ is compact, therefore $\exists R > 0$ such that $\partial u(\Gamma) \subset B(0, R)$. Let $x_1, x_2 \in \Gamma$, we want to show that $|u(x_2) - u(x_1)| \leq C|x_2 - x_1|$. By definition of subdifferentials,

$$u(x) \geq u(x_1) + \langle \alpha_1, x - x_1 \rangle$$
$$u(x) \geq u(x_2) + \langle \alpha_2, x - x_2 \rangle$$

for all $x \in D$, and with $\alpha_i \in \partial u(x_i) \subset B(0, R)$. Taking $x = x_2, x_1$ respectively in the above two inequalities,

$$R|x_2 - x_1| \geq \langle \alpha_1, x_1 - x_2 \rangle \geq u(x_1) - u(x_2)$$
$$R|x_2 - x_1| \geq \langle \alpha_2, x_2 - x_1 \rangle \geq u(x_2) - u(x_1)$$

where we used that $|\alpha_i| \leq R$. Taking R as constant C above, the conclusion follows. □

Corollary Non-differentiability set Nondiff(u) of a convex function u is a Lebesgue null set.

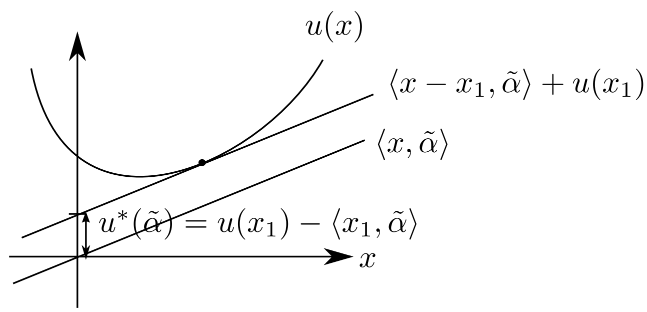
Proof: This follow from Rademacher's theorem which states that Lipschitz function is differentiable almost everywhere.

We will need the definition of Legendre transform:

Definition For u convex, the **Legendre transform** is a function u^* given by

$$u^*(\alpha) = \sup_x [\langle \alpha, x \rangle - u(x)]$$

Note in particular that u^* is also convex since it is pointwise supremum of convex (actually affine) functions $\alpha \mapsto \langle \alpha, x \rangle - u(x)$ thus locally Lipschitz by previous step.



By last corollary it suffices to show

$$B \subset \text{Nondiff}(u^*)$$

The following two observations are easy to check:

- If $\alpha \in \partial u(x_1)$ then the supremum in definition of $u^*(\alpha)$ is attained at $x = x_1$.
- Assume $\alpha \in B$, $\alpha \in \partial u(x_1) \cap \partial u(x_2)$ with $x_1 \neq x_2$, then by previous observation

$$u^*(\alpha) = \langle \alpha, x_1 \rangle - u(x_1) = \langle \alpha, x_2 \rangle - u(x_2)$$

Let $F(x) := \langle x, \alpha \rangle - u(x)$, then F is concave and achieves maximum at 2 distinct points, it follows that F is constanton $tx_2 + (1-t)x_1$ for $0 \leq t \leq 1$.

Claim: If $\alpha \in \partial u(x_1) \cap \partial u(x_2)$, then u^* is not differentiable at α.

Proof: $\alpha \in \partial u(x_1)$ and $\alpha \in \partial u(x_2)$ implies $x_1, x_2 \in \partial u^*(\alpha)$, therefore u^* is not differentiable at α. □

As a corollary we now have $B \subset \text{Nondiff}(u^*)$ which is a null set by previous Corollary, this finishes the proof for Claim 2. □.

With Claim 1, $\text{MA}(u)(E)$ is well-defined for any subset $E \subset D$, to prove the theorem we will also need to check that,

- $\text{MA}(u)(E) \geq 0$ for all E. This is obvious since Lebesgue measure is nonnegative

- $\mathrm{MA}(u)(\emptyset) = 0$. This is also trivial.
- $\mathrm{MA}(u)$ has countable additivity, that is for any countable disjoint collection of subsets $\{E_j\}$ we have

  $$\sum_j \mathrm{MA}(u)(E_j) = \mathrm{MA}(u)\left(\bigcup_j E_j\right)$$

  this also follow from Claim 2 since although the sets whose Lebesgue measure defines Monge–Ampère measure may overlap, the overlaps are null sets. We have

  $$\mathrm{MA}(u)\left(\coprod_j E_j\right) = m\left(\bigcup_j \bigcup_{x\in E_j} \partial u(x)\right) \le \sum_j m\left(\bigcup_{x\in E_j} \partial u(x)\right) = \sum_j \mathrm{MA}(u)(E_j)$$

  $$\begin{aligned}\mathrm{MA}(u)\left(\coprod_j E_j\right) &= m\left(\bigcup_j \bigcup_{x\in E_j} \partial u(x)\right) \ge m\left(\bigcup_j \bigcup_{E_j} \partial u(x) - X\right) = m\left(\bigcup_j \bigcup_{x\in E_j} (\partial u(x) - X)\right)\\ &= m\left(\coprod_j \bigcup_{x\in E_j} (\partial u(x) - X)\right) = \sum_j m\left(\left(\bigcup_{x\in E_j} \partial u(x)\right) - X\right)\\ &\ge \sum_j \left(m\left(\bigcup_{x\in E_j} \partial u(x)\right) - m(X)\right) = \sum_j m\left(\bigcup_{x\in E_j} \partial u(x)\right) = \sum_j \mathrm{MA}(u)(E_j),\end{aligned}$$

  where

  $$X := \{\alpha \in \mathbb{R}^n | \exists x_1 \ne x_2 \in \operatorname{dom} u, \alpha \in \partial u(x_1) \cap \partial u(x_2)\}$$

  is the set of subdifferentials coming from more than one points in $\operatorname{dom}(u)$ (i.e. domain of u), it is then clear that

  $$\bigcup_{x\in E_j} (\partial u(x) - X) \text{ and } \bigcup_{x\in E_i} (\partial u(x) - X)$$

  are disjoint and that the set X is measure zero since it is a subset of the non-differentiable locus of a finite convex function (i.e., u^*).

This completes the proof of the Theorem. □

Theorem 2.8 [1] For $\Omega \subset \mathbb{R}^n$ a bounded strictly convex domain, $g : \partial\Omega \to \mathbb{R}$ continuous, the Dirichlet problem for homogeneous Monge-Ampère equation

$$\begin{cases} \mathrm{MA}(u) = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

has unique solution $u \in \mathrm{Cvx}(\Omega) \cap C^0(\bar{\Omega})$.

Sketch of proof: Define

$$u = \sup\{a : a \text{ affine in } \Omega, a \le g \text{ on } \partial\Omega\}.$$

Note since u is defined as supremum of a family of convex functions, it is itself convex and we can talk about subdifferentials of u. Next we will see that for this upper envelope and any point in interior of domain, there is a point on the boundary that has the same subdifferential.

Claim Let u be the upper envelope defined above, then we have

$$\partial u(\Omega) \subset \partial u(\partial\Omega).$$

Proof: Let $\alpha \in \partial u(x_0)$ for some $x_0 \in \Omega$. Define

$$a(x) := \langle \alpha, x - x_0 \rangle,$$

note that this is an affine function in x. Since Ω is bounded, $g(x) - a(x)$ attains minimum at some $x_1 \in \partial\Omega$. Observe that $x \mapsto a(x) + g(x_1) - a(x_1)$ is affine and is $\leq g$ on $\partial\Omega$ thus by definition of upper envelope function u we have $u(x) \geq a(x) + g(x_1) - a(x_1)$ but two sides agree when $x = x_1 \in \partial\Omega$ thus $\alpha \in \partial u(x_1) \subset \partial u(\partial\Omega)$. □

To show $u \geq g$ on the boundary we will need some barrier function. Geometrically u is the lower boundary of the convex hull of the graph of boundary value function $\{(x, g(x)) | x \in \partial\Omega\}$

Lecture 3: The Cauchy problem for the homogeneous real Monge-Ampere equation. The Legendre transform in more detail. Convex hulls and the double Legendre dual: regularity and basic properties

We will move on to Cauchy problem of homogeneous Monge–Ampére equation which is more motivated in physics but also more complicated and less studied. First consider boundary value problem:

$$\begin{aligned}
\mathrm{MA}(u) &= 0 \quad \text{ in } [0,1] \times \mathbb{R} \to \mathbb{R} \\
u(0, \cdot) &= u_0 \in C^\infty \cap \mathrm{SCvx}, \quad u_i : \mathbb{R} \to \mathbb{R} \\
u(1, \cdot) &= u_1 \in C^\infty \cap \mathrm{SCvx}
\end{aligned}$$

where we specified some Dirichlet data at $t = 0, t = 1$. We have shown above that $\mathrm{MA}(u) = 0$ iff $\mathrm{Im}(\partial u)$ is an Lebesgue nulls set. It turns out that the solution fills in between u_0 and u_1 with a surface of zero curvature and gives a foliation of the product domain $[0,1] \times \mathbb{R}$ with lines on which unit normal remain a constant vector. We can give an analytic expression to the solution using partial Legendre transform defined as

$$u_t^* = u^*(t, y) = \sup_x \left[\langle x, y \rangle - u(t, x)\right]$$

where $u_t(x) := u(t, x)$. To interpret $\mathrm{Im}(\partial u)$ in terms of partial Lebesgue transform, we need some useful tricks. Below we assume derivative exists wherever derivative is taken. We have

$$\begin{aligned}
\frac{\partial u^*}{\partial t}(t, y) &= \frac{\partial}{\partial t}\left[\langle (\nabla u_t)^{-1}(y), y \rangle - u(t, (\nabla f_t)^{-1}(y))\right] \\
&= -\frac{\partial u}{\partial t}(t, (\nabla u_t)^{-1}(y))
\end{aligned}$$

where we assume that $u_t \in C^1$ and strictly convex therefore ∇u_t is always invertible and we used the fact that $(\nabla u_t) \circ (\nabla u_t)^{-1} = \mathrm{id}$. For $u \in C^1$ this allow us to express the subdifferential of u as

$$\begin{aligned}
\partial u(t, x) &= \left\{ \left(\frac{\partial u}{\partial t}(t, x), \nabla u_t(x) \right) \right\} \\
&= \left\{ \left(-\frac{\partial u^*}{\partial t}(t, (\nabla u_t)(x)), \nabla u_t(x) \right) \right\}
\end{aligned}$$

therefore the subdifferential sets will lie inside graphs of functions $\dot{u}_t^*$ where $\cdot$ stands for time derivative, i.e.

$$\begin{aligned}
\mathrm{Im}(\partial u)\big|_{\{t\} \times \mathbb{R}^n} &= \mathrm{gr}\left(-\frac{\partial u_t^*}{\partial t} \right)\bigg|_{\mathrm{Im}(\nabla u_t)} \\
\mathrm{Im}(\partial u) &= \bigcup_t \mathrm{Im}\, \partial u_t \bigg|_{\{t\} \times \mathbb{R}^n}
\end{aligned}$$

which suggests that we look for functions u such that $\dot{u}_t^*$ is actually constant in t in which case all the subdifferentials will lie in graph of a single function. If instead the graphs of $\dot{u}_t^*$ fill up a 2-dimensional region, it is possible that Monge-Ampere measure will charge it with a positive number. To be more precise, if $u \in C^2$ and strictly convex, we have the following formula of second variation of Legendre transform:

$$\begin{aligned}
\left.\frac{\partial^2 u_t^*}{\partial t^2}\right|_{(t,y)} &= -\left.\frac{\partial^2 u_t}{\partial t^2}\right|_{(t,x)} + \langle \nabla_x \left.\frac{\partial u_t}{\partial t}\right|_{(t,x)}, \nabla_y \left.\frac{\partial u_t}{\partial t}\right|_{(t,y)} \rangle \\
&= -\left.\frac{\partial^2 u_t}{\partial t^2}\right|_{(t,x)} - \langle \nabla_x \frac{\partial u_t}{\partial t}, \left(\nabla_x^2 u_t\right)^{-1} \nabla_x \frac{\partial u_t}{\partial t} \rangle
\end{aligned}$$

where $y = \nabla u_t(x)$. RHS vanishes if u solves HRMA. To see this we need a bit of linear algebra, for block matrix with D invertible we have

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(D) \det(A - BD^{-1}C)$$

therefore assuming $\nabla_x u \neq 0$ HRMA can be reduced to

$$\det \begin{pmatrix} \ddot{u}_t & \nabla_x \dot{u}_t \\ (\nabla_x \dot{u}_t)^T & \nabla_x^2 u_t \end{pmatrix} = \ddot{u}_t \det \nabla_x^2 u_t - \langle (\nabla_x^2 u_t)^* \nabla_x \dot{u}_t, \nabla_x \dot{u}_t \rangle$$

where $*$ means adjoint matrix. We can divide by $\det \nabla_x^2 u \neq 0$ to get

$$\ddot{u} - \langle (\nabla_x^2 u)^{-1} \nabla_x \dot{u}, \nabla_x \dot{u} \rangle = 0$$

using the first and second variation formula above we can show that partial Legendre transformation linearizes the equation, giving

$$u_t^*(y) = (1-t)u_0^* + tu_1^*$$

where we can take another partial Legendre transform to get

$$u(t,x) = ((1-t)u_0^* + tu_1^*)^*(x)$$

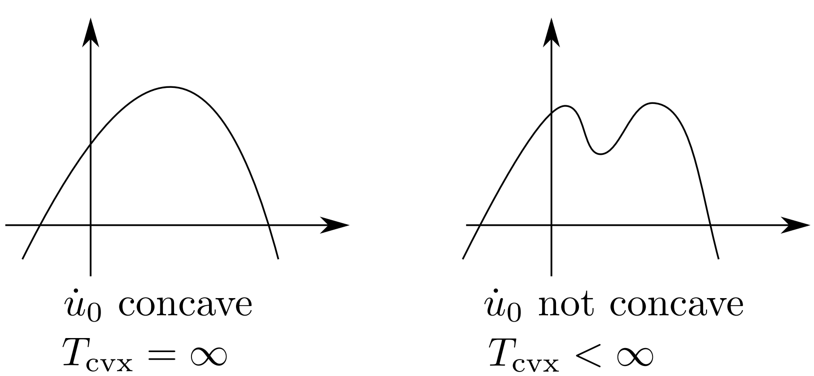
Now let us consider initial value problem on $[0,T] \times \mathbb{R}^n$:

$$\begin{aligned}
\mathrm{MA}(u) &= 0 \\
u(0,\cdot) &= u_0 \\
\frac{\partial u}{\partial t}(0,\cdot) &= \dot{u}_0
\end{aligned}$$

where $\dot{u}_0, u_0$ are given. Previous computation tells us we should expect solution of the form $u_t^*(y) = u_0^* - t\dot{u}_0 \circ (\nabla u_0)^{-1}$ but we need to be more careful: in the boundary value case, all functions are assumed to be C^2 and strictly convex so $\nabla_x^2 u$ is always invertible while in IVP this is no longer true. Actually we can define a maximal existence time

$$T_{\text{cvx}} := \sup\{t > 0 : u_0^* - t \cdot u_0 \circ (\nabla u_0)^{-1} \text{ is convex}\}$$

which can be shown to be positive but is not always infinity. With linearization under partial Legendre transform it is not hard to see that if $\dot{u}_0$ is concave, then $T_{\text{cvx}} = \infty$, otherwise $T_{\text{cvx}} < \infty$.



In the following we will state a theorem about solution of this IVP where we switch our notation and recall the form of solution to be

$$\psi(t,x) = \left(\psi_0^* - t\dot{\psi}_0 \circ (\nabla\psi_0)^{-1}\right)^*$$

where we are interested in application to Kahler geometry which will give some assumptions on ψ_0 and $\dot{\psi}_0$, in particular $\mathrm{Im}\,\nabla\psi_0 = P$ will be a compact convex polytope and $\dot{\psi}_0$ is assumed to be bounded on that polytope.

Theorem 1 [2]

1. ψ solves HRMA on the regular locus of ψ (i.e. where ψ is C^1) In particular, on $[0, T_{\mathrm{cvx}}] \times \mathbb{R}^n$
2. If $T > T_{\mathrm{cvx}}$, ψ fails to sovle HRMA. Futhermore, $\mathrm{MA}(\psi)$ whose total mass is positive has an apriori upper bound given by
$$\int_{[0,T]\times\mathbb{R}^n} \mathrm{MA}\,\psi \leq \mathrm{Vol}\left(\mathrm{epi}(-\dot{u}_0)\backslash\, \mathrm{epi}(-(\dot{u}_0)^{**})\right)$$
where epi stand for epigraph

Open Question: Does there exist a solution with poorer regularity? (It is shown in [2] that there does not exist any C^1 solution)

Corollary If a solution exists, it must be $C^{0,1}$ but not C^1.

Conjecture Such a solution does not exist.

Lecture 4: Obstruction to the solution of the Cauchy problem for the homogeneous real Monge-Ampere equation: upper and lower bounds on the subdifferential and strict convexity of the Legendre subsolution

We will continue the sketch of the proof that

$$\psi(s,x) := \left(\psi_0^* - s\dot{\psi}_0 \circ (\nabla\psi_0)^{-1}\right)^*$$

does not solve HRMA but does solve the equation on regular locus

Fact 1 $\psi_s(x) := \psi(s,x)$ is strictly convex on $\mathbb{R}^n \Leftrightarrow \psi_s^*(y) \in C^1(P\backslash\partial P)$.
Recall that

$$u_s(y) := u_0 + s\dot{u}_0$$

where $u_0 = \psi_0^*$, $\dot{u}_0 = -\dot{\psi}_0 \circ (\nabla\psi_0)^{-1}$ and recall the underlying assumption $\nabla_x\psi_s \in P$ a fixed closed compact convex polytope with $\overline{\mathrm{Im}\,\nabla_x\psi_s} = P$. There is a standard result in convex analysis: a function is strictly convex iff its dual f^* is C^1 away from boundary. (c.f. Rockafellar)

Lemma 4.1

(i) $\psi_s^* = u_s^{**}$

(ii) Let Δ_{n+1} be the unit simplex in $\mathbb{R}^{n+1}$ defined by

$$\Delta_{n+1} = \{\lambda = (\lambda_1, \ldots, \lambda_{n+1} \in \mathbb{R}^{n+1} : \lambda_i \geq 0, \sum \lambda_i = 1\}$$

then we have

$$u_s^{**}(y) = \inf\{\lambda \cdot (u_s(y_1), \ldots, u_s(y_{n+1})) : \lambda \in \Delta_{n+1}, y_i \in P, \sum \lambda_i y_i = y\}$$

Claim:

$$(y, u_s^{**}(y)) = \sum_i^{n+1} \lambda(y_i, u_s(y_i))$$

that is, in particular, the infimum in representation formula of the biconjugate function is actually attained.

Proof: We know that by convexity and since $\text{epi}(u_s^{**}) = \text{co}(\text{epi}(u_s))$, that

$$(y, u_s^{**}(y)) = \sum_{i=1}^{m} \lambda_i(y_i, r_i)$$

with $r_i \geq u_s(y_i)$, but since $\sum_i \lambda_i(y_i, u_s(y_i))$ also belongs to epigraph of u_s, we have $r_i = u_s(y_i)$ for all $i \in I$ (see below), since otherwise the point $(y, u_s^{**}(y))$ will lie directly above another point in convex hull of $\text{epi} u_s$.

Now by a corollary to Caratheodory's theorem we can take $m = n + 1$ $\Box$

Definition Let $I = \{i : \lambda_i > 0\}$. We say that $y_i, i \in I$ are *called upon* by y.

Lemma $f^{***} = f^*$ for any continuous function f.

Claim $y_i \in P \backslash \partial P$

Proof: The proof relies on the formula

$$\partial u_s^{**}(y) = \bigcap_{i \in I} \partial u_s(y_i)$$

To show this, take $\delta \in \partial u_s^{**}(y)$, by definition of subdifferential we have

$$u_s^{**}(z) \geq u_s^{**}(y) + \langle z - y, \delta \rangle \quad \forall z$$

equivalently

$$u_s^{***}(\delta) + u_s^{**}(y) = \langle \delta, y \rangle$$

(where if δ is not necessarily from subdifferential, equality would have been replaced by $\geq$) Using the fact that $f^{***} = f^*$ as f^* is already convex, this is equivalent to

$$u_s^*(\delta) + \sum_i \lambda_i u_s^{**}(y_i) = \langle \delta, \sum \lambda_i y_i \rangle$$

since $u_s^*(\delta) + u_s(y_i) \geq \langle \delta, y_i \rangle$, we have

$$u_s^*(\delta) + u_s(y_i) = \langle \delta, y_i \rangle$$

for all $i \in I$, therefore $\delta \in \partial u_s(y_i)$ for all $i \in I$. This finishes the proof of the formula.

Now the LHS of the formula is non-empty since $y \in P \backslash \partial P$ and each $\partial u_s(y_i)$ is a singleton, therefore the formula implies that all $\partial u_s(y_i)$ has to be the same singleton, proving that u_s^{**} is indeed C^1. $\Box$

Claim Let $y \in P \backslash \partial P$, $x := \nabla u_s^{**}(y)$, then

$$\text{co}\{(-\dot{u}_0(v), v) : v \in \gamma_x \psi(s, x)\} \subset \partial \psi(s, x) \subset \text{co}\{(-\dot{u}_0(v), v) : v \in \partial_x \psi(s, x)\}$$

where $\mathrm{co}\, A$ is convex hull of A and

$$\partial_x \psi(s,x) := \partial \psi_s(x)$$
$$\gamma_x \psi(s,x) := \gamma \psi_s(x)$$

and $\gamma f(x)$ is the *reachable subdifferential* of f at x defined by

$$\gamma f(x) := \{\delta : \exists \{x_k\} \subset \text{regular locus of } f \text{ s.t. } \delta = \lim_k \nabla f(x_k)\}$$

Fact (Lemma 2.1) $\partial f(x) = \mathrm{co}\, \gamma f(x)$

Proof: **Second inclusion**: Let $\tilde{v} \in \gamma\psi(s,x)$ by the above $\partial\psi(s,x) = \mathrm{co}\, \gamma\psi(s,x)$ and since taking convex hull perserves inclusion, it suffices to show

Claim: $\gamma\psi(s,x) \subset \{(-\dot{u}_0(v), v) : v \in \partial_x \psi(s,x)\}$

Proof: By definition there is $(s_k, x_k) \in$ regular locus of ψ, such that

$$\tilde{v} = \lim_k \nabla\psi(s_k, x_k) = \lim_k (-\dot{u}_0 \circ \nabla_x \psi_{s_k}(x_k), \nabla_x \psi_{s_k}(x_k))$$

Lecture 5: Obstruction to the solution of the Cauchy problem for the homogeneous real Monge-Ampere equation: completion of the proof of the main theorem. Subequations: definition

Recall that $MA(\psi) \neq 0$ if $\psi = (u_0 + s\dot{u}_0)^*$ where $u_0 = \psi_0^*$ and $\dot{u}_0 = -\dot{\psi}_0 \circ (\nabla\psi_0)^{-1}$ with $\psi_0 \in C^\infty \cap \mathrm{SCvx}(\mathbb{R}^n)$, (where SCvx stands for strictly convex functions) $\mathrm{Im}\nabla\psi_0 = P$, a compact convex polytope, $\dot{\psi}_0 \in C^\infty \cap L^\infty(\mathbb{R}^n)$ with $T > T_{cvx} = \inf\{s > 0 : (u + s\dot{u}_0)^{**} \neq (u + s\dot{u}_0)\}$. We want to show

$$\int_0^T \int_{\mathbb{R}^n} MA(\psi) > 0$$

Lemma 12.4 Let $y \in P \backslash \partial P$, and set $x := \nabla u_s^{**}(y)$. Then

$$\mathrm{co}\{(-\dot{u}_0(v), v) v \in \gamma_x \psi(s,x)\} \subset \partial\psi(s,x) \subset \mathrm{co}\{(-\dot{u}_0(v), v) : v \in \partial_x \psi(s,x)\}$$

This provides us upper and lower bounds of the subdifferential $\partial\psi(\{s\} \times \mathbb{R}^n)$. This does not quite imply that MA measure charges positively once $T > T_{cvx}$ but it does imply the a priori upper bound of MA mass:

$$\int_0^T \int_{\mathbb{R}^n} MA(\psi) \leq \mathrm{Vol}(\mathrm{epi}(-\dot{u}_0) \backslash \mathrm{epi}(-(\dot{u}_0)^{**}))$$

which is a partial regularity statement. This also raises the question of whether ψ is an optimal solution in the sense that for arbitrary subsolution η of Cauchy problem

$$\int_0^\infty \int_{\mathbb{R}^n} MA(\eta) \geq \int_0^\infty \int_{\mathbb{R}^n} MA(\psi) > 0.$$

Two sets play an important role in the proof of main theoerm of [2]

Definition

$$A_s = \{y \in P : u_s(y) \neq u_s^{**}(y)\},$$
$$A'_s = \{y \in P \backslash \partial P : \exists v \neq y \text{ with } \nabla u_s^{**}(y) = \nabla u_s(y)\}.$$

Note that the two sets coincide for generic function. The proof will be tremendously more complicated if they are not the same and we won't go into these details, so we will assume

$$A_s \backslash \partial P = A'_s$$

This is indeed ture for generic data $u_0, \dot{u}_0$. Now on A_s we can generate positive mass as soon as the left hand side in Lemma 12.4, i.e. $\mathrm{co}\{(-\dot{u}_0(v), v) : v \in \gamma_x \psi(s,x)\}$ contains more than one point, it will contain the line segment in between, this will be where the function is not differentiable. We have

$$\int MA(\psi) = \int_{\Sigma_{sing}} MA(\psi)$$

where Σ_{sing} is the singular set defined not by non-C^1 locus of ψ but by

$$\Sigma_{reg}(T) = \bigcup_{s \in [0,T]} \{s\} \times \partial u_s(\mathrm{int}(P \backslash (\partial P \cup A'_s)))$$

and $\Sigma_{sing}(T) = [0,T] \times \mathbb{R}^n \backslash \Sigma_{reg}(T)$. We partition the singular set by some equivalence classes of the polytope given by

$$Q(s,y) := \{v \in P : \nabla u_s^{**}(v) = \nabla u_s^{**}(y)\}$$

for $y \in P \backslash \partial P$. The partition of A_s is give by

$$\overline{A_s} \backslash \partial P = \bigcup_{v \in A_s} Q(s,y) \cap (\overline{A_s} \backslash \partial P)$$

define

$$\tilde{Q}(s,y) := \mathrm{co}\{(-\dot{u}_0(v), v) : v \in \gamma_x \psi(s, \nabla u_s^{**}(y))\}$$

which lies above $Q(s,y)$. We will show that this set is 'big'. Intuitively the graph of $-\dot{u}_0$ is not concave above the set Q. Define

$$U(s,y) = \pi_p(\tilde{Q}(s,y) \backslash \mathrm{gr}(-\dot{u}_0)\big|_{Q(s,y)})$$

which is contained in $Q(s,y)$, we have

Lemma 13.2 $U(s,y)$ is a set with non-empty interior relative to $Q(s,y)$

Final step is to put everything together. For this we need a continuity result whose proof is easy

Lemma 8.2 $s \mapsto A_s$ is continuous as a set valued map.

Intuitively when s increases, the set where u_s is non-convex grows continuously. On a given time slice

$$\partial \psi(\{s\} \times \mathbb{R}^n) \supset \bigcup_{v \in A_s \backslash \partial P} \tilde{Q}(s,v)$$

now

$$\tilde{U}(s,y) := \tilde{Q}(s,y) \cap \pi_p^{-1}(U(s,y))$$

is affine and does not intersect the graph of $-\dot{u}_0$ over $U(s,y)$. Now the conclusion that MA charges positive mass follows from a Fubini type argument.

Dirichlet Duality

In this section we introduce the concept of subequations which is useful in the study of a large family of equations from geometry depending only on Hessian of the unknown function. We consider equation the form

$$\mathcal{F}(\nabla^2 u) = 0$$

where $\mathcal{F} : Sym^2(\mathbb{R}^n) \subset \mathbb{R}^{n^2} \to \mathbb{R}$ is a continuous function on the space of symmetric $n \times n$ matrices. We wish to study weak solution to this equation. For some choices of $\mathcal{F}$ there will be a natural notion of weak solution. For this we associate to the function $\mathcal{F}$ a subset $F \subset Sym^2(\mathbb{R}^n)$ such that for $A \in Sym^2(\mathbb{R}^n)$ one has $\mathcal{F}(A) = 0$ if $A \in \partial F$, i.e., $\partial F \subset \mathcal{F}^{-1}(0)$. Note there is in general no formula to derive the subset F from given equation $\mathcal{F}$, since the later is in general not unique.

Example Let $\mathcal{F} = \mathrm{tr}$, the corresponding equation of hessian is just the Laplacian equation:

$$\Delta u = 0$$

and for the associated subset we can take

$$F = \{A \in Sym^2(\mathbb{R}^n) : \mathrm{tr}\, A \geq 0\}$$

We consider the Dirichlet problem of equation

$$\begin{cases} \mathcal{F}(\nabla^2 u) = 0 & \text{in } \Omega \\ u = f & \text{on } \partial\Omega \end{cases}$$

To get guarantee both uniqueness and existence of solution for this equation, we need some assumption for $\mathcal{F}$ or equivalently for the subset F.

Definition (Definition 3.1 of [3]) A proper nonempty closed subset $F \subset Sym^2(\mathbb{R}^n)$ is a **Dirichlet set** (or a **subequation** if it satisfies the **positivity condition**

$$F + \mathcal{P} \subset F$$

where

$$\mathcal{P} \equiv \{A \in Sym^2(\mathbb{R}^n) : A \geq 0\}$$

denotes the set of nonnegative quadratic forms on $\mathbb{R}^n$

Definition Suppose $F \subset Sym^2(\mathbb{R}^n)$ is a Dirichlet set. The **Dirichlet dual** of F is the set

$$\tilde{F} = \sim (-\,\mathrm{Int}\, F) = -(\sim \mathrm{Int}\, F)$$

where $\sim$ stands for complement and $-$ just takes the negative matrices

Fact: The subequation condition can be characterized with dual condition

$$A \in F \Leftrightarrow A + B \in \tilde{\mathcal{P}} \;\; \forall B \in \tilde{F}$$

Lecture 6: Subequations: basic properties. Subaffine functions. Functions of type F. Comparison with viscosity solutions

Recall again we are studying the Dirichlet problem

$$\begin{cases} \mathcal{F}(\nabla^2 u) = 0 & \text{on } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

The general idea of studying such problem is find a set $F \subset Sym^2(\mathbb{R}^n)$ such that $A \in \partial F$ and $\mathcal{F}(A) = 0$ and to construct a set of functions that will be thought of as subsolutions/competitors. Once we have constructed this class, one can define $u := \sup(\text{subsolutions})$ and prove u solves the Dirichlet problem, in other words $\mathcal{F}(\nabla^2 u) = 0$ and it realizes boundary condition. For such a construction to work we will need

w_1, w_2 subsolutions $\Rightarrow \max\{w_1, w_2\}$ also a subsolution, that is the class of subsolutions to be closed under taking maximum.

Definition 4.4 An u.s.c function u on X (some bounded domain in $\mathbb{R}^n$) is of type F (we write $u \in F(X)$) if

$$u + v \in SA(X), \quad \forall v \in C^2(X) \cap F(X)$$

Definition $u \in C^2(X)$ is of type F if $\nabla^2 u(x) \in F$ for all $x \in X$.

Definition $u \in SA(X)$ called **subaffine** functions if $\forall K \subset X$ compact and a affine in K, we have maximum principle of the form

$$u \leq a \text{ on } \partial K \Rightarrow u \leq a \text{ on } K$$

When one generalize a class of funtion in the study of fully nonlinear PDE, it is wise to first look at convex functions, in other words one should first check if the technique works for Monge-Ampère equation. The concept of subaffine functions and dual Dirichlet set is thus motivated by:

Proposition 2.5-2.6

$$u \in Cvx(X) \Leftrightarrow u + v \in SA(X), \ \forall v \in SA(X)$$
$$u \in SA(X) \Leftrightarrow u + v \in SA(X), \ \forall v \in Cvx(X)$$

Fact (Lemma 2.2) $u \notin SA(X)$ iff $\exists x_0 \in X$, a affine, $\epsilon > 0$ such that

$$(u - a)(x) \leq -\epsilon |x - x_0|^2 \text{ near } x_0$$
$$u(x_0) = a(x_0)$$

Consider $SA(X)$, the class of subaffine functions on X and $P(X) = Cvx(X)$ the class of convex functions on X. These two classes of functions each characterize the other one, there is clearly an underlying duality between them. Thus we wish to define a notion of duality such that $\tilde{P}(X) = SA(X)$ and $\tilde{SA}(X) = P(X)$. It is easier if we start on the level of matrices, what allows one to do that is the following theorem due to Slodkowski ([4] Prop 2.4). Note that for $F \subset Sym^2(\mathbb{R}^n)$ we defined

$$\tilde{F} = (-\operatorname{int} F)^c$$

it follows from definition that $\tilde{\tilde{F}} = F$. (Note Slodkowski's theory is more complicated where double dual need not equal to original class while triple dual is always the same with single dual) In order to ensure solution of Dirichlet problem some kind of ellipticity is called form, it comes in the form of our positivity condition

$$F + \mathcal{P} \subset F$$

(Note if one take arbitrary, for instance some hyperbolic equation, Dirichlet problem may very well be ill-posed)

Lemma 4.3 $u \in C^2(X) \cap F(X)$ iff $u + v \in SA(X)$ for all $v \in C^2(X) \cap \tilde{F}(X)$

Lecture 7: Subequations: rays sets and boundary defining functions

For Dirichlet problem given by

$$\begin{cases} \mathcal{F}(\nabla^2 u) = 0 & \text{in } \Omega \\ u = f & \text{on } \partial\Omega \end{cases}$$

Similar to Perron solution to Dirichlet problem of Laplacian equation, we will eventually construct solution of the type $u = \sup_{g \in F(\Omega)} g$ and we will need to make some geometric assumption on the boundary to guarantee that the Perron solution is continuous up to the boundary. Note that there are examples where the Perron solution fails to be continuous up to the boudary, e.g. for HRMA when boundary is not strictly convex.

Theorem 5.12 [3] F is a subequation, then $\partial\Omega$ is strictly $\vec{F}$-convex iff $\exists$ global defining function $\rho \in C^\infty(\bar{\Omega})$ for $\partial\Omega$ which is strictly of type $\vec{F}$ on $\bar{\Omega}$.

First we need to define some concepts in this theorem

Definition A smooth function $\rho \in C^\infty(\bar{\Omega})$ is called a **global defining function** for $\partial\Omega$ if $\Omega = \{\rho < 0\}$ and $\nabla\rho \neq 0$ on $\partial\Omega$.

Definition 5.8 [3]

$$\vec{F} := \{A \in Sym^2(\mathbb{R}^n) : tA \in F \text{ for } t \gg 1\}$$

is the **Dirichlet ray associated to** F a given subequation.

Definition 5.1 [3] $\partial\Omega$ is strictly $\vec{F}$-convex if

$$\nabla^2\rho(x)\Big|_{T_x\partial\Omega} = B\Big|_{T_x\partial\Omega} \tag{**}$$

for some $B \in \operatorname{int}\vec{F}$. $\forall x \in \partial\Omega$ and for some $\rho \in C^\infty$ defined on a neighborhood U_x of $x \in \mathbb{R}^n$ such that $U_x \cap \Omega = \{\rho < 0\} \cap U_x$, $\nabla\rho\Big|_{U\cap\partial\Omega} \neq 0$. Such ρ is called a **local defining function**

Examples Consider the Laplacian equation

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = f & \text{on } \partial\Omega \end{cases}$$

let $F = \{A \in Sym^2(\mathbb{R}^n) : \operatorname{tr} A \geq 0\}$ thus $\vec{F} = F$ and $\operatorname{int}\vec{F} = \{A : \operatorname{tr} A > 0\}$. Any smooth doain is strictly $\vec{F}$-convex (*)

Exercise 1 For any smooth Ω (i.e. with smooth $\partial\Omega$) there is a global defining function (hint: use partition of unity argument) such that

$$\nabla^2\rho(x)\Big|_{T_x\partial\Omega} = B\Big|_{T_x\partial\Omega}$$

Exercise 2 Show (*), i.e. that any smooth domain is strictly $\vec{\mathcal{P}}$-convex as above

We know that when $\partial\Omega$ is smooth Dirichlet problem is solvable and solution is unique. We will see that similar conclusion holds for general subequation.

Lemma 5.2 Definition 5.1 is independent of choice of ρ.

Corollary 5.4 $\partial\Omega$ is strictly $\vec{F}$-convex at $x \in \partial\Omega$ iff $II(x) = B\Big|_{T_x\partial\Omega}$ for some $B \in \operatorname{int}\vec{F}$, where II is the second fundamental form of $\partial\Omega$ as a smooth hypersurface.

Idea of proof: Choose the signed distance function

$$\rho = \delta(x) = \begin{cases} -\operatorname{dist}(x, \partial\Omega) & x \in \Omega \\ \operatorname{dist}(x, \partial\Omega) & x \notin \Omega \end{cases}$$

Exercise 3 Orthogonally decompose $\mathbb{R}^n = \mathbb{R}\nabla\delta(x) \oplus (\nabla\delta(x))^\perp$, then

$$\nabla^2\delta(x) = \begin{pmatrix} 0 & 0 \\ 0 & II(x) \end{pmatrix}$$

Corollary 5.10 $A \in \operatorname{int}\vec{F}$ iff $\exists\epsilon > 0, R > 0$ such that $C(A - \epsilon I) \in F$ for all $c \geq R$.

Moreover, $\exists\epsilon, R > 0$ such that

$$C(\rho - \frac{\epsilon}{2}|x|^2) \in F(\bar{\Omega}), \ \ \forall c \geq R$$

Note the left hand side produces a subsolution (function of type F) whose behaviour on $\partial\Omega$ is very simple, just like barrier function in Perron method for Laplacian equation.

Sketch of Proof of Theorem 5.12: $\Leftarrow$ is trivial. For $\Rightarrow$ we proceed by two steps

1st Step : $\exists$ global defining function $\rho \in C^\infty(\bar{\Omega})$ (See exercise below)

2nd Step : Let $\tilde{\rho} = \rho + C\rho^2$. (Note: all local defining functions are locally equivalent and all are equivalent to signed distance function introduced earlier, adding a ρ^2 term is to force it to be strictly convex, see below). We have

$$\nabla^2\tilde{\rho} = (1 + 2C\rho)\nabla^2\rho + 2C\nabla\rho \circ \nabla\rho$$

where the 2nd term is a rank 1 matrix. If $x \in \partial\Omega$,

$$\nabla^2\tilde{\rho}(x) = \nabla^2\rho(x) + C\nabla\rho(x) \circ \nabla\rho(x) \in \operatorname{int}\vec{F} \qquad (***)$$

Claim (Lemma 5.3(ii)[3]) $\forall c \gg 1$, $\nabla^2\tilde{\rho}(x) \in \operatorname{int}\vec{F}$

Lemma 5.3 Let ρ be local defining function, n be unit normal vector field along $\partial\Omega$ then

$$(**) \Leftrightarrow \nabla^2\rho(x) + tn \circ n \in \operatorname{int}\vec{F}$$

for $t \gg 1$. Note $n \circ n$ is orthogonal projection onto $\mathbb{R}n$ and $\mathbb{R}^n = \mathbb{R}n \oplus (\mathbb{R}n)^\perp = \mathbb{R}n \oplus (T_x\partial\Omega)$.

Proof: $\Rightarrow$: Given B, by (**) we can decompose according to the direct sum $\mathbb{R}n \oplus T_x\partial\Omega$

$$\nabla^2\rho(x) - B = \begin{pmatrix} a & \alpha \\ \alpha^T & 0 \end{pmatrix}$$

hence

$$\nabla^2\rho(x) + tn \circ n = \underline{\nabla^2\rho(x) + tn \circ n + \epsilon I - B} + B - \epsilon I$$

$$\underline{\ldots} = \begin{pmatrix} t + a + \epsilon & \alpha \\ \alpha^T & \epsilon I \end{pmatrix}$$

and $B - \epsilon I \in \operatorname{int}\vec{F}$ since B is in the ray set and ϵ is small. We can take t large enough so that the underlined part is positive definite. Now the conclusion follows by the property of Dirichlet sets $\operatorname{int} F + \mathcal{P} \subset \operatorname{int} F$. $\square$

Exercise 4 Show that for t large enough the underlined matrix is indeed positive definite

Consider again (***). The constant $C = C(x)$ depends on x but since $\partial\Omega$ is compact we can always take a constant $C \gg 1$ such that (***) holds for all $x \in \partial\Omega$. We also take C such that this holds for $x \in U \supset \partial\Omega$ where U is a neighborhood of the boundary $\partial\Omega$.

3rd Step : Define $\hat{\rho}(x) = \max\{\rho(x), \frac{\delta|x|^2}{2} - r\}$, $\delta, r > 0$. This is not a smooth function, only Lipschitz. We choose r such that $U \supset \{x \in \mathbb{R}^n : -r < \rho(x) < r\}$. Note that $\hat{\rho} = \rho$ on U and where maximum function takes value equals to second function, Hessian is $\delta I \in \text{int}\,\vec{F}$.

4th Step : Smooth the maximum function $M(u_1, u_2) = \max\{u_1, u_2\}$ by defining a two variable function $M_\epsilon(t_1, t_2)$. This will finish the proof and provide a strictly $\vec{F}$-plurisubharmonic global defining function.

Lecture 8: Subequations: Proof of Main Theorem in Harvey-Lawson

Recall the F-Dirichlet problem:

$$\begin{cases} \mathcal{F}(\nabla^2 u) = 0 & \text{on } X \qquad (*) \\ u = g & \text{on } \partial X \end{cases}$$

and we have defined subequation $F \subset Sym^2(\mathbb{R}^n)$ such that for $u \in C^2 \cap \partial F(X)$ (*) holds. We call elements of $F(X)$ subsolutions and that of $-\tilde{F}(X)$ supersolutions.

Definition 6.1 $u \in F(X) \cap (-\tilde{F}(X))$ and $u\Big|_{\partial X} = g$, we say that u solves (*) **in the sense of Harvey-Lawson**

Main Theorem (Theorem 6.2 [3]) $\Omega \subset \mathbb{R}^n$ a bounded domain with smooth boundary, F a subequation, suppose

(i) $\partial\Omega$ is strictly-$\vec{F}$ convex

(ii) $\partial\Omega$ is strictly-$\vec{\tilde{F}}$ convex

then (*) admits a unique $C^0(\bar{\Omega})$ solution (i.e. continuous up to boundary and attain boundary value g)

Proof of Existence: Set

$$u(x) := \sup\{v(x) : v \in F(\Omega) \cap USC(\bar{\Omega}), v\Big|_{\partial\Omega} \leq g\}$$

(denote the set where supremum is over as $\mathscr{F}(\Omega)$) where $USC(\Omega)$ is important for making sense of the condition that v be controlled by g on the boundary.

Facts:

(1) $u \in USC(\Omega)$

(2) $u \in F(\Omega)$

(3) $u\Big|_{\partial\Omega} \leq g$

(4) $u = usc(u)$ where $usc(f)(x) := \text{limsup}_{y \to x} f(y)$ is upper semicontinuous regularization of u. This will be used to show (2)

(1') $usc(u) \in F(\Omega)$

(5) $usc(u)\Big|_{\partial\Omega} \le g + \delta$ for all $\delta > 0$ (This $\Rightarrow$(3).)

Note that (1)-(3) will imply that $u \in \mathscr{F}(\Omega)$ and (5)+(1')$\Rightarrow$(4). Observe that u is bounded from above: if $F \subset \tilde{\mathcal{P}} = SA$ then $F(\Omega)$ satisfies the maximum principle since the subaffine functions satisfy the maximum principle. However even if that does not hold, we always have for some λ such that $F + \lambda I \subset \tilde{\mathcal{P}}$ (where λ is a fixed number depending only on F). (To show this, it suffices to show the dualized version $\tilde{F} - \lambda I \supset \mathcal{P} \Leftrightarrow \tilde{F} \supset \mathcal{P} + \lambda I$ but this follows from $\tilde{F} \supset \mathcal{P} + \tilde{F}$, i.e. the positivity condition for $\tilde{F}$ and the fact that for large enough λ, $\lambda I \in \tilde{F}$.) Therefore for $v \in F(\Omega)$ we have

$$\exists \lambda \ \ v + \frac{1}{2}\lambda |x|^2 \in SA(\Omega)$$

Proof of (5): Assumption (ii) and theorem from last lecture guarantees existence of boundary defining function for Ω, $\rho \in \text{int}\, \overrightarrow{F}(\bar{\Omega})$. Let $x_0 \in \partial\Omega$, there exists $\epsilon > 0, R > 0$ such that

$$C(\rho - \epsilon|x - x_0|^2) \in \tilde{F}(\bar{\Omega}) \ \ \forall c \ge R$$

on $\partial\Omega$ we have

$$g + C(\rho - \epsilon|x - x_0|^2) = g - C\epsilon|x - x_0|^2 \le g(x_0) + \delta$$

where the last inequality holds if we take $C \ll 1$. Now let $v \in \mathscr{F}(\Omega)$ from the family and define

$$w := v + C(\rho - \epsilon|x - x_0|^2) \in F(\Omega) + \tilde{F}(\Omega) \subset SA(\Omega)$$

which will play similar role as barrier function in Perron's method. By maximum principal $\sup_{\bar{\Omega}} w = \sup_{\partial\Omega} w$ and since

$$\sup_{\partial\Omega} w \le g(x_0) + \delta$$

so for all v in the family

$$v + C(\rho - \epsilon|x - x_0|^2) \le g(x_0) + \delta$$

it follows that $usc(u)(x_0) \le g(x_0) + \delta$ □

We have already showed $u \le g$ on $\partial\Omega$ (Prop 6.7 [3]) To show that $u\Big|_{\partial\Omega} = g$ it suffices to show

Lemma (Lemma 6.9 [3]) If Ω has strictly $\vec{F}$-convex boundary, then

$$\liminf_{x \to x_0} u(x) \ge g(x_0) \ \ \forall x_0 \in \partial\Omega$$

Proof: Fix $\delta > 0$, let $\rho \in \text{int}\, \vec{F}(\bar{\Omega})$ be boundary defining function for Ω. $\exists \epsilon > 0$ such that for $C \gg 0$

$$C(\rho - \epsilon|x - x_0|^2) \in F(\Omega)$$

For $C \gg 0$ we have on $\partial\Omega$

$$g(x_0) + C(\rho - \epsilon|x - x_0|^2) \le g(x) + \delta$$

Let

$$v(x) := g(x_0) - \delta + C(\rho - \epsilon|x - x_0|^2)$$

we have $v \in \mathscr{F}$ thus $v \le \sup \mathscr{F}(\Omega) = u \in F(\Omega)$,

$$u \ge g(x_0) - \delta + C(\rho - \epsilon|x - x_0|^2)$$

so

$$\liminf_{x \to x_0} u(x) \ge \liminf_{x \to x_0} v(x) = g(x_0) + \delta \ \ \forall \delta > 0$$

□

Next we will show interior continuity and uniqueness in a more general context [5].

Lecture 9: Solving the Dirichlet problem for domains with non-smooth boundary: the main theorem

We prove a more general version of the result in [3] for Dirichlet Problem with weak boundary assumptions

Theorem 7.8 [5] F a subequation, $X \subset \mathbb{R}^n$ bounded domain, $\bar{X}$ a manifold with embeded corners. φ is a consistent continuous function on ∂X with

(i) ∂X is strictly (F, φ)-convex

(ii) ∂X is strictly $(\tilde{F}, -\varphi)$-convex

Then $\exists! u \in C^0(\bar{X})$ s.t. $u \in F(X)$, $-u \in \tilde{F}(X)$, $u\Big|_{\partial X} = \varphi$.

Definitions Given X manifold with corners, ∂X is also a manifold with coners equipped with a map $\iota_X : \partial X \to X$ which could be non-injective. We say that X has embeded corners if ∂X is a disjoint union of finitely many open and closed subsets on which ι_X is injective. $\varphi \in C^0(\partial X)$ is **consistent** if it is constant on fibres of ι_X.

Examples $[0, 1] \times D$ with D a closed manifold is a manifold with corners. Another example is taking $D = [0, 1]$ then $X = [0, 1] \times [0, 1]$. Boundary ∂X has four disjoint components. An example of a manifold with corners whose corner is not embeded is provided by the teardrop.

Now we need to define boundary convexity (which as we can see above in our context also depends on boundary data!)

Definition 7.1 Write $\partial X = \coprod_i \partial X_i$. ∂X_i is strictly $\vec{F}$-convex if

$$\mathrm{II}|_{T_x \partial X_i} = B|_{T_x \partial X_i} \text{ for some } B \in \operatorname{int} \vec{F} \ \forall x \in \partial X_i$$

Definition 7.5 Let $\varphi \in C^0(\partial X)$ be consistent $u \in F(X)$ is called a **subsolution for** (F, φ)**-Dirichlet problem** if $u|_{\partial X} \leq \varphi$

Definition 7.6 ∂X is strictly (F, φ) if ∂X can be decomposed to $A \coprod B$ with A, B unions of boundary components such that

(i) $\forall p \in A, \forall \delta > 0$, $\exists C^0(\bar{X})$ subsolution of the (F, φ)-Dirichlet problem that is δ-maximal at p

(ii) B is strictly $\vec{F}$.

For a subsolution of the F-Dirichlet problem for (X, g), i.e. a function $u \in F(X) \cap USC(X)$ such that $u \leq g$ on ∂X is called δ**-maximal** for some $\delta > 0$ at x_0 if $u(x_0) \geq g(x_0) - \delta$. The assumption (i) is natural in view of the following lemma:

Lemma 7.9 Suppose $\partial X_i \subset B$, then there is a w satisfying (i) above

Proof: Choose boundary defining function ρ for ∂X_i, we have showed before that there is $\epsilon, R > 0$ such that $C(\rho - \epsilon|x|^2) \in \tilde{F}(X)$. Since we can add any affine function and still stay in $F(X)$ we have

$$C(\rho - \epsilon|x - x_0|^2) \in \tilde{F}(X)$$

given $\delta > 0$ there is $C \gg 0$ such that

$$-g(x) + C(\rho(x) - \epsilon|x - x_0|^2) \leq -g(x) - C\epsilon|x - x_0|^2 \leq -g(x_0) + \delta$$

let

$$w = C(\rho(x) - \epsilon|x - x_0|^2) + g(x_0) + \delta$$

It is easy to verify that this w has the property we wanted. $\square$

Definition 7.2 $\rho \in C^\infty(\bar{X})$ global defining function for ∂X_i if

$$\rho|_{\bar{X}\setminus\partial X_i} < 0 \quad \rho|_{\partial X_i} = 0 \quad \nabla\rho|_{\partial X_i} \neq 0$$

Note we only require ρ to vanish on the component ∂X_i

The following proposition will be proven next time:

Proposition (Prop 7.3 [5]) For ∂X_i strictly $\vec{F}$-convex, there exists global defining function ρ for ∂X_i that is strictly $\vec{F}$-convex, in particular $\exists \epsilon, R > 0$ such that

$$C(\rho - \epsilon|x|^2) \in F(\bar{X}), \;\; \forall C \geq R$$

Proof of Theorem 7.8: Set

$$\mathscr{F}(\varphi) = \{v \text{ subsolution for } (F, \varphi)\text{-Dirichlet Problem}\}$$

It is easy to see that this set is nonempty since we can always take functions of the form $C_1|x|^2 - C_2$ with C_1, C_2 appropriately chosen. Let

$$u := \sup\{v : v \in \mathscr{F}(\varphi)\}$$

This is well-defined since we can show that the family $\mathscr{F}$ is uniformly bounded from above, $v \leq C = C(X, \varphi, F)$. By [3] $usc(u) \in F(X)$. We now need to show $usc(u) \leq \varphi$ on ∂X. Either using condition (ii) of the theorem or by Lemma 7.9, $\exists w \in C^0(\bar{X})$ subsolution for $(\tilde{F}, -\varphi)$-Dirichlet Problem that is δ-maximal at $p \in \partial X_i$ for all i, p, δ, that is $w(p) \geq -\varphi(p) - \delta$ and $w \in \tilde{F}(X)$, therefore $w \leq -\varphi$ on ∂X. Let $v \in \mathscr{F}(\varphi)$ we have $v + w \in SA(X)$ and $v + w \leq 0$ on ∂X. By Maximum Principle of subaffine functions $v + w \leq 0$ on X, thus $v \leq -w$ and $u \leq -w$ since u is the upper envelope of v's. Since w is continuous, taking upper-semi-continuous regularization on both sides we have $usc(u) \leq -w$. Therefore $usc(u)(p) \leq -w(p) \leq \varphi(p) + \delta$ for all $\delta > 0$, it follows that $usc(u) \leq \varphi$.

The interior continuity of upper envelope will be proven next time. $\square$

Lecture 10: Solving the Dirichlet problem for domains with non-smooth boundary: boundary defining functions for boundary components

Recall from last lecture we have sketched the proof of Theorem 7.8 [5] for domains with non-smooth boundary with a more general condition of boundary convexity. The crucial step is existence of a δ maximal subsolution w which is constructed by Lemma 7.9 using Proposition 7.3. We now prove this proposition:

Proposition 7.3 Boundary component ∂X_i is strictly $\vec{F}$-convex, then there exists global defining function $\rho \in C^\infty(\bar{X})$ for ∂X_i that is strictly of type $\vec{F}$
Remark Note we treat boundary components separately since conclusion of the above proposition is in general false! For example let $X = [0,1] \times [0,1]$ and $g(s,t) = s(s-1)t(t-1)$ and $F = \mathcal{P}$ thus F-convexity is ordinary convexity. It is not hard to see that each corner is a saddle point of g thus it is not convex. Any other global defining function must also has this property since they must vanish on four sides, be negative inside the square and have nonvanishing normal derivative on interior of each side.

Proof: X can be covered with atlas (see [6]), i.e. a collection of maps $\psi_i : U_i \to X$ where U_i are open sets in $\mathbb{R}^n_+ = [0, \infty)^n$

By Whitney extension theorem there exists smooth extension of ψ_i (also denoted ψ) to $\tilde{U}_i$ open in the whole space $\mathbb{R}^n$ which is still diffeomorphism to its image $W_i := \psi_i(\tilde{U}_i)$. By definition of embeded corners, there is an l such that $\psi_i^{-1}(\partial X_i \cap W_i) = \{x \in U_i, x_l = 0\}$, it is easy to see $-x_l$ is a nice local defining function, set

$$f_i = -x_l \circ \psi_i^{-1} \text{ on } W_i$$

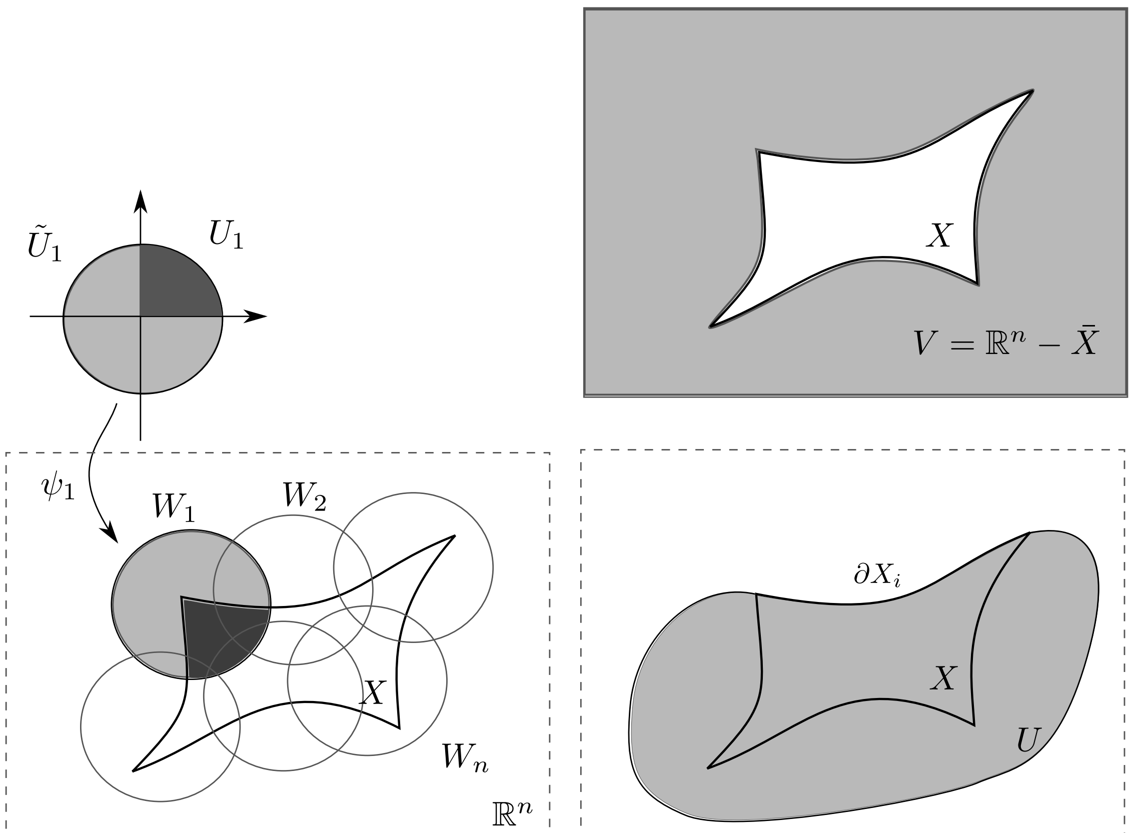
Now let U be an open set whose intersection with $\bar{X}$ is $\bar{X}$ with the boundary piece ∂X_i removed.

$$\{W_i \text{ for all } i, U, V = \mathbb{R}^n - \bar{X}\}$$

form an open cover of $\mathbb{R}^n$, let $\alpha_i, \alpha_U, \alpha_V$ be a smooth partition of unity subordinary to it, i.e. they are supported on each open set and $\sum_i \alpha_i + \alpha_U + \alpha_V \equiv 1$ and set

$$\rho := \sum \alpha_i f_i + \alpha_V - \alpha_U \in C^\infty(\mathbb{R}^n)$$

it is straightforward to check that (i) $\rho = 0$ on ∂X_i, (ii) $\rho < 0$ on $\bar{X} \backslash \partial X_i$, (iii) $\nabla \rho \neq 0$ on ∂X_i



Now we can construct a global defining function using similar steps as in [3], first note $\tilde{\rho} = \rho + C\rho^2$ is strictly $\vec{F}$ local defining function on a neighborhood of ∂X_i for C sufficiently large, then use partition of unity argument to modify it so it is negative on $\bar{X} \backslash \partial X_i$ that is strictly $\vec{F}$ in a neighborhood W of ∂X_i. Choose $r > 0$ small such that $\{\rho > -r\} \cap \bar{X}$ is contained in W, and $\delta > 0$ small such that $\delta|x|^2 - r < 0$ on ∂X_i, $\hat{\rho} = \max\{\rho, \delta|x|^2 - r\}$ agrees with ρ in a neighborhood of ∂X_i where ρ is strict $\vec{F}$ and equals the quadratic function elsewhere, this will give the desired global defining function after a smoothing process similar to [3]. $\square$

The last step is to show interior continuity of the upper envelope function u of the family $\mathscr{F}(\varphi)$. For this we use an argument due to Walsh [7].

Proposition (Prop 6.11 [3]) Let u be the upper envelope defined above, then $u \in C(\bar{\Omega})$

Proof: Let $\Omega_\delta = \{\text{dist}(x, \partial\Omega) > \delta\}$ and $C_\delta = \{\text{dist}(x, \partial\Omega) < \delta\}$ be the δ-collar of the boundary and let $u_y = u(x+y)$ be y-translate of u where we set the undefined part equal to $-\infty$. By continuity of u, there is $\delta > 0$ such that $u_y < u + \epsilon$ on $C_{2\delta}$ for $|y| \leq \delta$. We claim

$$u_y \leq u + \epsilon \text{ on } \bar{\Omega}$$

as well, for $|y| \leq \delta$. It is easy to see that conclusion follow from this claim. Note first that by translation property and affine property of Dirichlet sets $u_y - \epsilon \in F(\Omega_\delta)$. Since $u_y < u + \epsilon$ on collar $C_{2\delta}$ we have

$$g_y := \max\{u_y - \epsilon, u\} \in F(\Omega)$$

by maximum property. But since g_y takes value u on collar, we have in fact g_y is in the family $\mathscr{F}(\varphi)$ hence $g_y \leq u$ on $\bar{\Omega}$ and the claim is proved. □

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